Aalto University
School of Business

Session 6:
Vehicle routing problems
35E00750 Logistics Systems and Analytics

Dr. Tri M. Tran
Assistant Professor of Operations Management
University of Groningen
https://www.rug.nl/staff/tri.tran/

Learning objectives

university of groningen
faculty of economics and business

- **Understand the Traveling Salesman Problem (TSP) formulations**

- **Understand Vehicle Routing Problem (VRP) formulations**

After this lecture, VRP will not be a "black box"

Slide courtesy of Dr. Ilke Bakir, Associate Professor, Department of Operations, Faculty of Economics and Business, University of Groningen

Aalto University
School of Business

Course contents

Part 1. Background

1. Understanding supply chains
2. Achieving supply chain fit
3. Mathematical programming for Logistics & SCM
4. Guest lecture: Janne Kilpua

Part 2. Transportation

5. Urban logistics
6. **Vehicle routing problems**

Part 3. Facilities

7. Warehousing technologies
8. Guest lecture: Vesa Hämetvaara (Konecranes)
9. Facility location problems

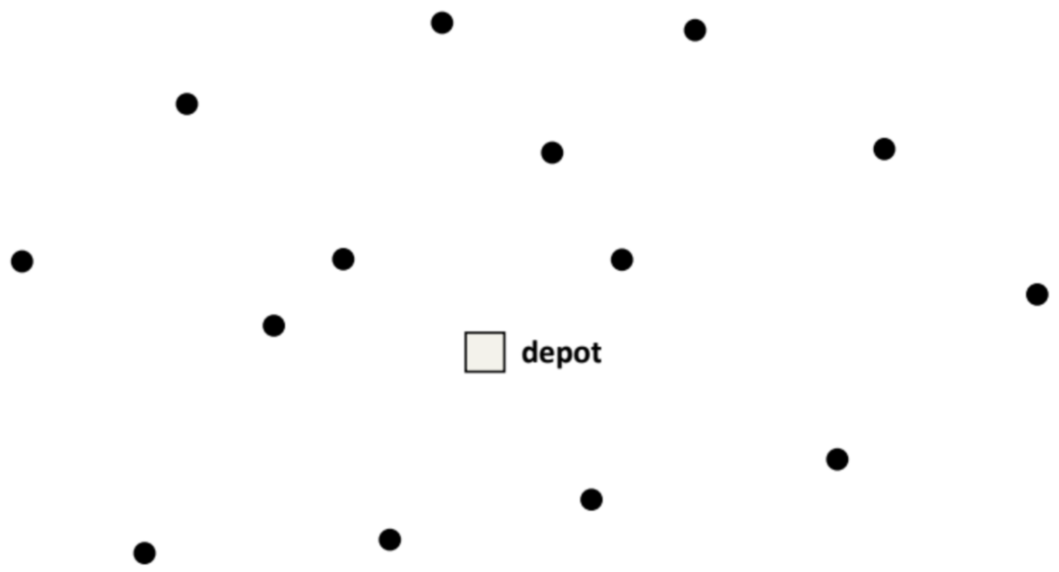
Part 4. Data

10. Digital logistics
11. Logistical drivers and metrics

Vehicle Routing Problems (VRP)

university of groningen
faculty of economics and business

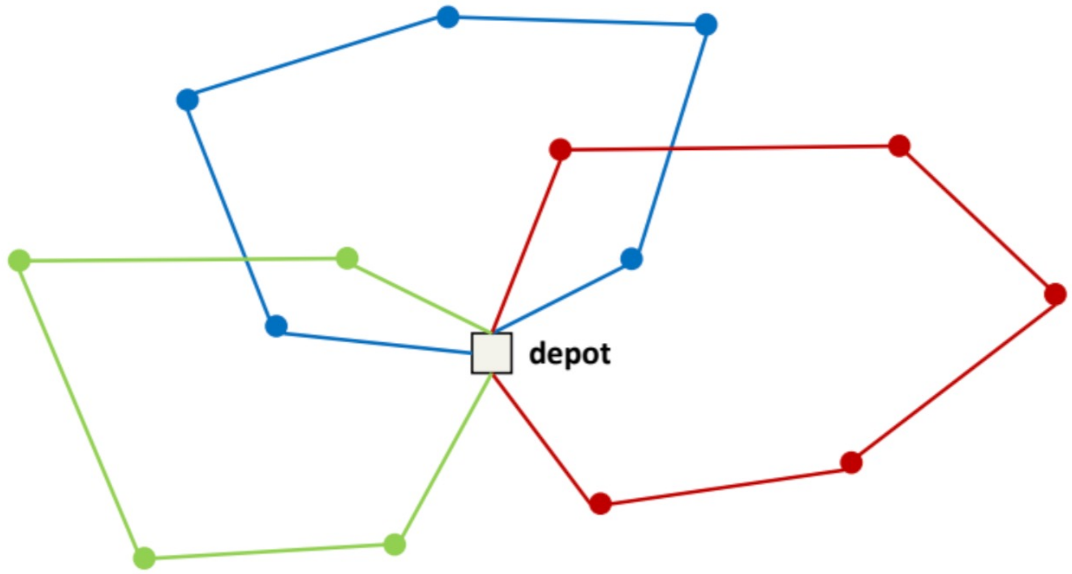
The problem of constructing routes from/to one or several depot(s) to visit a number of geographically dispersed locations to pick up or deliver goods



Vehicle Routing Problems (VRP)

university of groningen
faculty of economics and business

The problem of constructing routes from/to one or several depot(s) to visit a number of geographically dispersed locations to pick up or deliver goods



Vehicle Routing Problems (VRP)

The problem of constructing routes from/to one or several depot(s) to visit a number of geographically dispersed locations to pick up or deliver goods

So that:

- Each city is visited exactly once by exactly one vehicle
- All routes start and end at a depot

Single Vehicle Route: Traveling Salesman Problem (TSP)

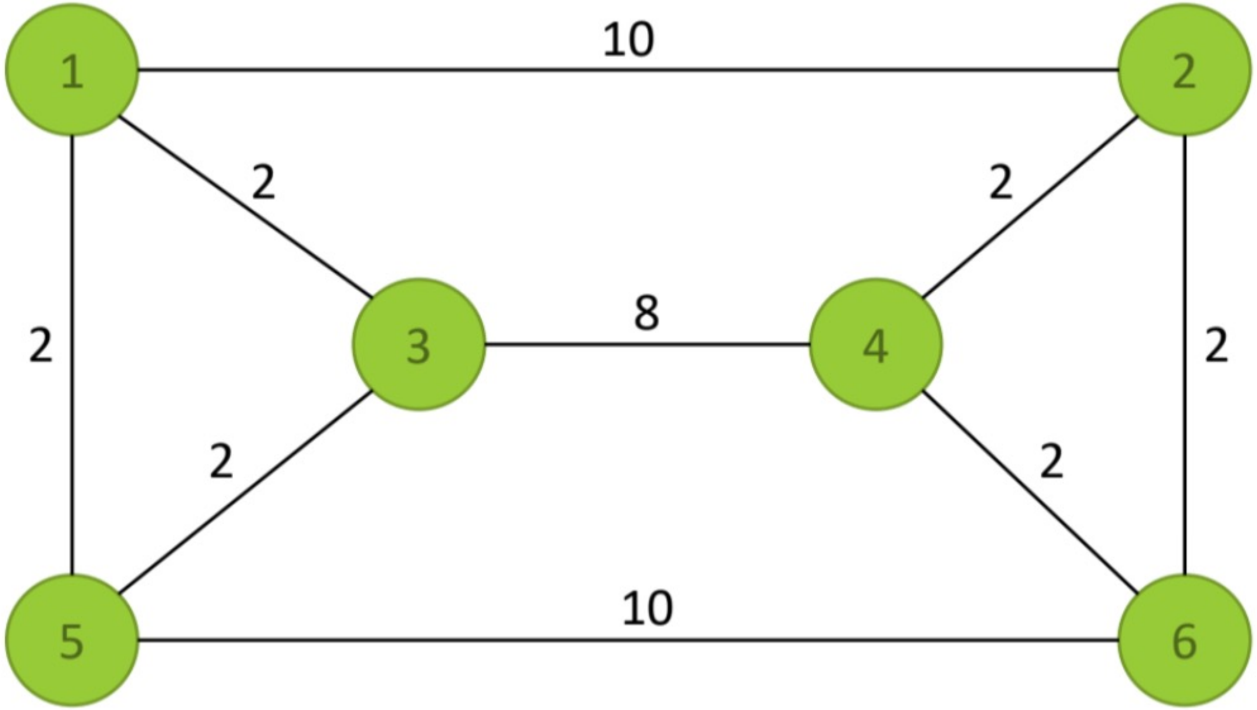
- **The Traveling Salesman Problem (TSP) seeks a route with a minimum total length visiting every point in a given set exactly once.**

- **More formal description:**
  - *"Suppose a salesman would like to visit several cities. Find the shortest route that visits each city exactly once and returns the salesman back to where they started"*

TSP Networks

university of groningen

faculty of economics and business

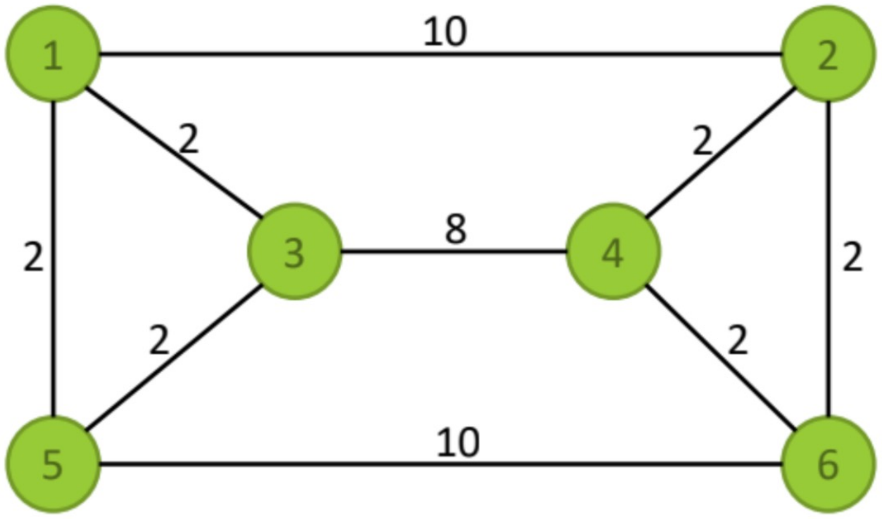


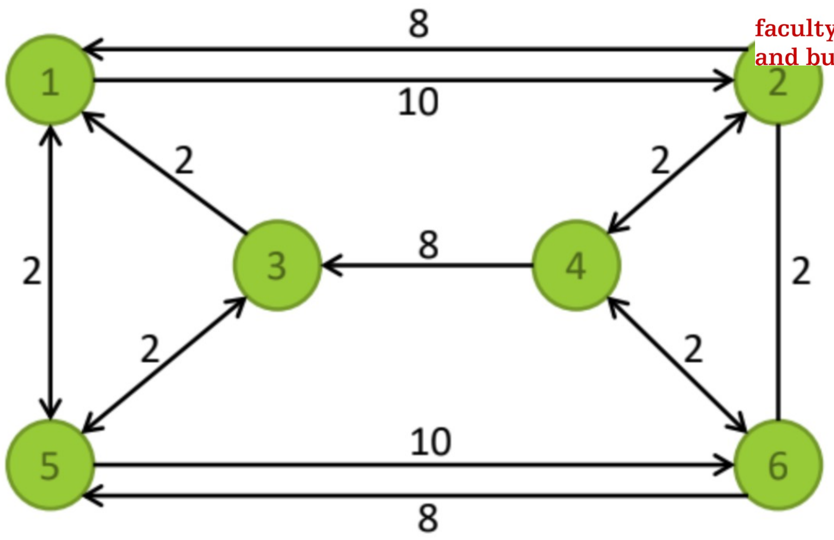
Aalto University School of Business

TSP Networks

university of groningen

faculty of economics and business





Symmetric TSP

- The distance between all pairs of two cities (i,j) is the same in both direction

Asymmetric TSP

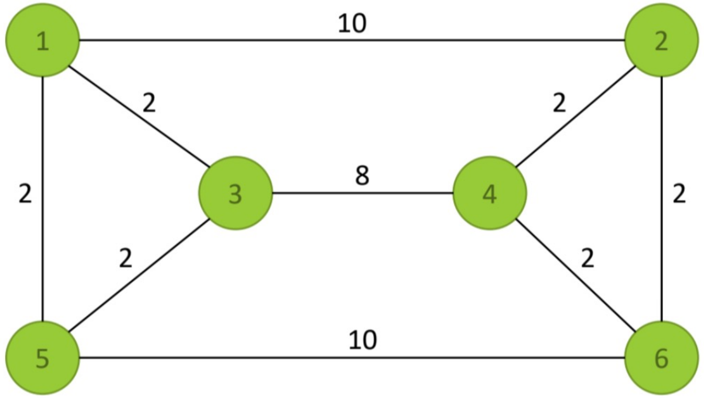
- Distances may differ between the *i-j* and *j-i* direction
- Paths may not exist in both directions *i-j* and *j-i*

Aalto University
School of Business

TSP: LP Formulation (Variables and Parameters)

university of groningen
faculty of economics and business

- **What are the variables?**
  - If tour uses the arc from city i to city j
  - $x_{ij} = 1$ if your passes from city i to city j
  - $x_{ij} = 0$ otherwise
- **Indicate the valid range of the variables**
  - x_{ij} are binary (0 or 1) variables
  - Denoted as $x_{ij} \in \{0, 1\} \; \forall i, j$
- **Parameters**
  - N: set of nodes
  - d_{ij}: Travel distance of arc (i, j)



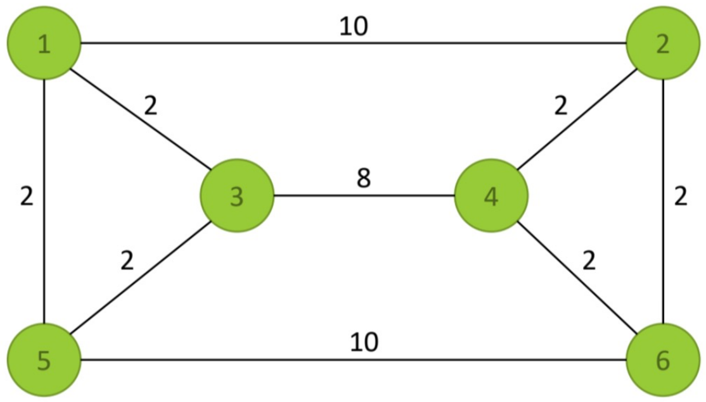
Aalto University
School of Business

TSP: LP Formulation (Objective FUNCTION)

- **Objective function**
  - Minimize route length
  - Minimize $10x_{12} + 2x_{13} + 2x_{15} + 10x_{21} + 2x_{24} + 2x_{26} + 2x_{31} + 2x_{35} + 8x_{34} + 2x_{42} + 8x_{43} + 2x_{46} + 2x_{51} + 2x_{53} + 10x_{56} + 2x_{62} + 2x_{64} + 10x_{65}$
  - Or in summation form:

$$Min \sum_{i \in N} \sum_{j \in N} d_{ij} x_{ij}$$

where d_{ij} denotes the length of the arc between city i and city j



TSP: LP Formulation (Constraints)

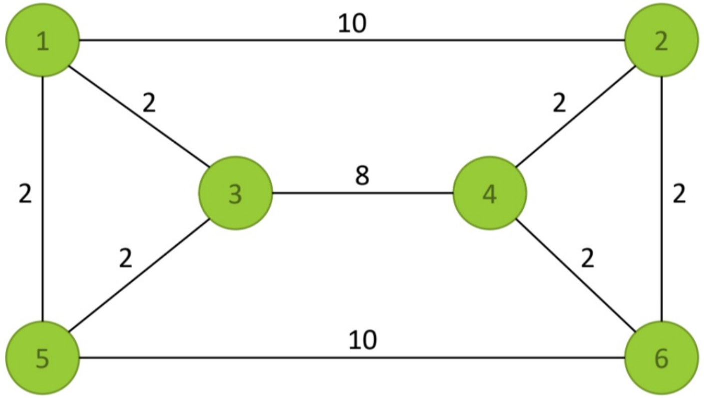
university of groningen
faculty of economics and business

- **Formulate the constraints**
  - For each city i, there is exactly **one** city that is its predecessor:

$$\sum_{j} x_{ji} = 1, \forall i \in N$$

  - For each city i, there is exactly **one** city that is its successor:

$$\sum_{j} x_{ij} = 1, \forall i \in N$$



TSP: LP Formulation (Constraints)

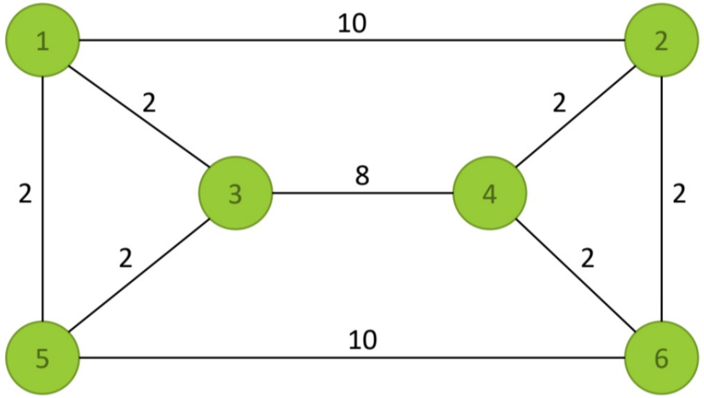
- **Example (for the given network):**

$$\sum_{j \in N} x_{ji} = 1, \forall i \in N$$

$$x_{21} + x_{31} + x_{51} = 1$$
$$x_{12} + x_{42} + x_{62} = 1$$
$$x_{13} + x_{43} + x_{53} = 1$$
$$x_{24} + x_{34} + x_{64} = 1$$
$$x_{15} + x_{35} + x_{65} = 1$$
$$x_{26} + x_{46} + x_{56} = 1$$

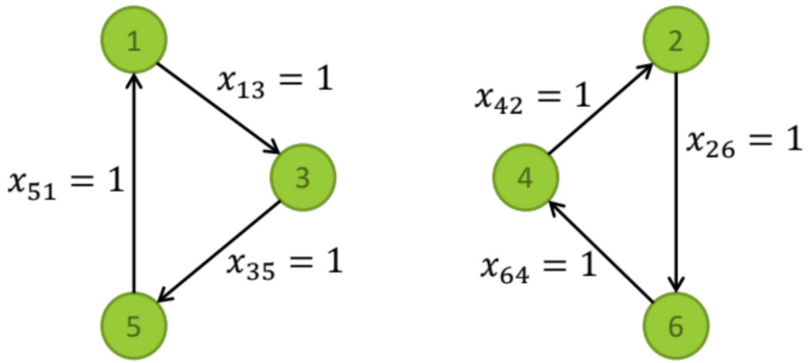
$$\sum_{j \in N} x_{ji} = 1, \forall i \in N$$

$$x_{12} + x_{13} + x_{15} = 1$$
$$x_{21} + x_{24} + x_{26} = 1$$
$$x_{31} + x_{34} + x_{35} = 1$$
$$x_{42} + x_{43} + x_{46} = 1$$
$$x_{51} + x_{53} + x_{56} = 1$$
$$x_{62} + x_{64} + x_{65} = 1$$



TSP: LP Formulation (Constraints)

- **So far, we have introduced two sets of constraints**
  - Every node i must have a predecessor node
  - Every node i must have a successor node
- **Now, consider this solution**

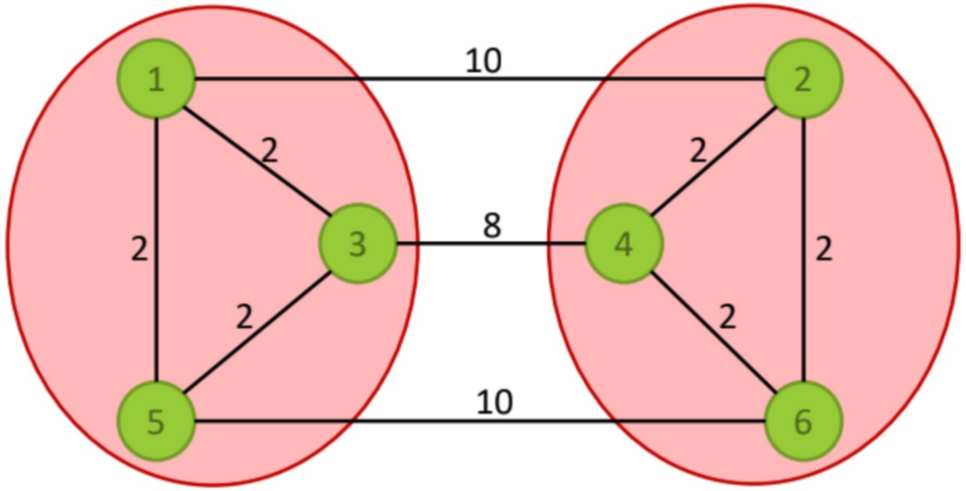


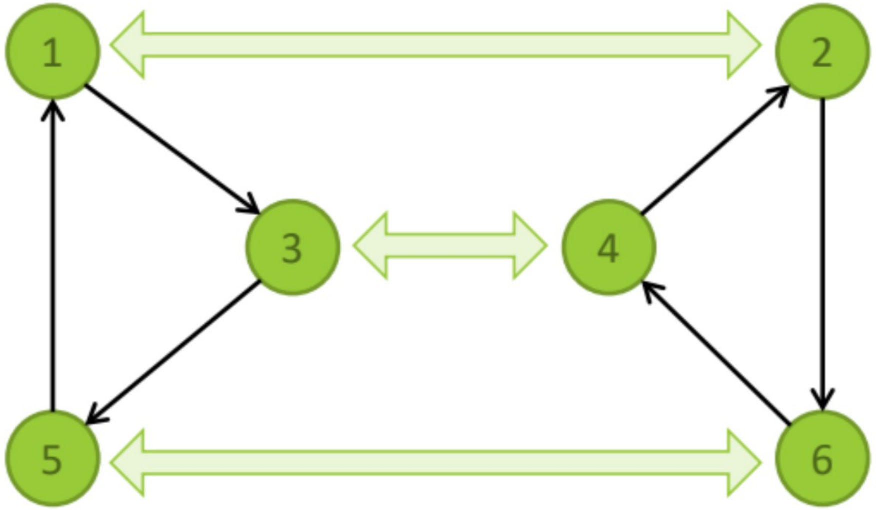
- It satisfies the two sets of constraints
  - *Every node has exactly one predecessor and exactly one successor*
- But it is NOT a valid TSP tour!
  - ***Remember****: TSP tour must be a route, i.e., the salesman must be able to start at one of the cities, visit all other cities, and come back to the city where they started.*
- So, we need more constraints.

TSP: LP Formulation (SECs)

university of groningen
faculty of economics and business

- We need **subtour elimination constraints** (SECs) to prevent solutions that contain subtours
  - At least one arc that leaves one of the nodes 1, 3, 5 and enters one of the nodes 2, 4, or 6 must be in the solution.
  - In this network, only arcs 1-to-2, 3-to-4, and 5-to-6 apply
- Mathematically
  - $x_{12} + x_{34} + x_{56} \geq 1$

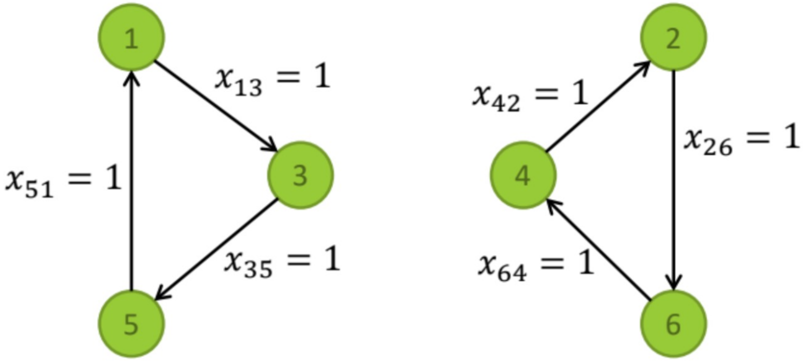




Aalto University
School of Business

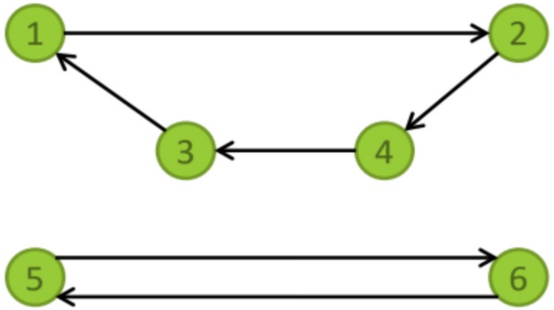
TSP: LP Formulation (SECs)

- Once we add the constraint $x_{12} + x_{34} + x_{56} \geq 1$ to the formulation, this solution

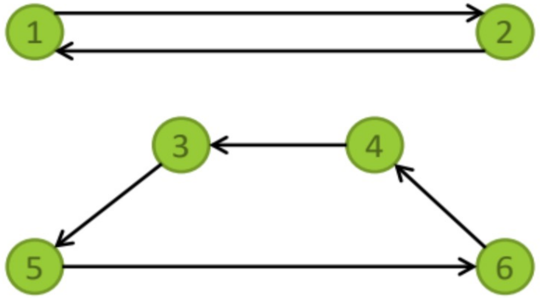


is not feasible anymore.

- But what about this one



- Or this one



- These solutions satisfy all the constraints defined so far, including $x_{12} + x_{34} + x_{56} \geq 1$. But they are still NOT valid TSP tours.

TFC: LP Formulation (SECs)

university of groningen
faculty of economics and business

- **The SECs need to be defined so that they prevent ALL POSSIBLE SUBTOURS!**
  - To guarantee, this, we need to define SECs for every node subset.
  - Remember:
    - *N is the set of all nodes (cities) in the network.*
    - *When we write $S \subset N$, it means that S is a **subset** of set N.*
      - For example, if $N = \{1, 2, 3, 4\}$ and $S = \{1, 2\}$, then $S \subset N$.
    - *SECs defined mathematically as follows.*

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1, \qquad \forall S \subset N: 2 \leq |S| \leq |N| - 1$$

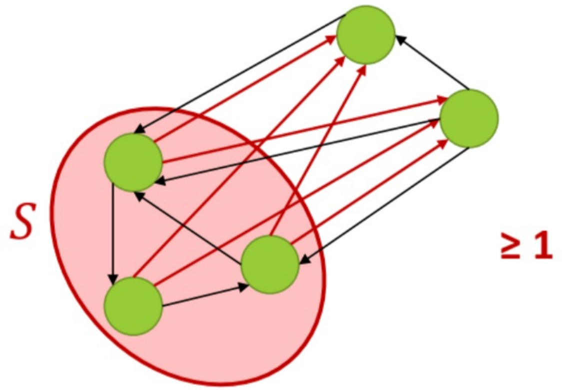
There must be at least one arc leaving a node in S and entering a node outside of S. These constraints must be defined for all subsets S that contain at least two nodes (because a node subset with a single node cannot be a subtour) and at most the number of nodes minus 1 node.

TSP: LP Formulation (SECs)

- For every node subset S such that $2 \leq |S| \leq |N| - 1$,

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1$$

must hold.



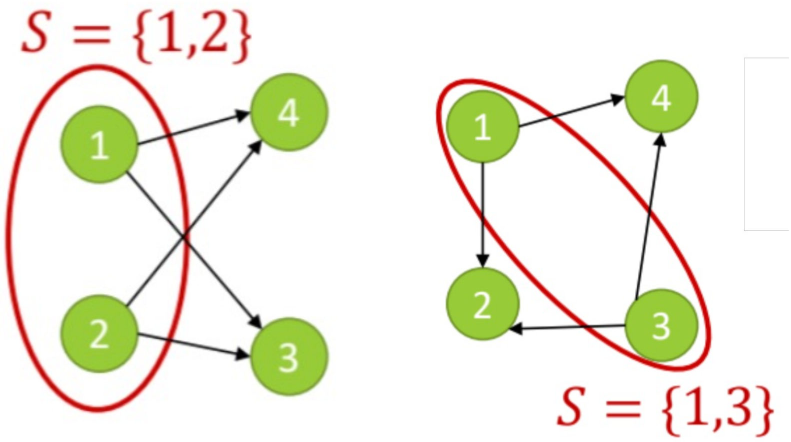
Recall: Subsets

- **As an example, let $N = \{1, 2, 3, 4\}$.**
  - This set $\boldsymbol{N}$ has 4 elements, and therefore it has subsets containing 0, 1, 2, 3, and 4 elements
    - *O-element subset: ∅ {empty set = a set with no elements in it}*
    - *1-element subsets: {1}, {2}, {3}, and {4}*
    - *2-element subsets: {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, and {3,4}*
    - *3-element subsets: {1,2,3}, {1,2,4}, {1,3,4}, and {2,3,4}*
    - *4-element subset: {1,2,3,4} (This is the set itself. Each set is a subset of itself.)*
  - A set with 4 elements has, in total, $2^4 = 16$ subsets.
  - We want the subsets of $\boldsymbol{N}$ with at least 2 and at most $|\boldsymbol{N}| - \mathbf{1} = \mathbf{3}$ elements for the SECs (specified by the expression $\forall S \subset N: 2 \leq |S| \leq |N| - 1$).
    - *Thus, we want 2-element and 3-elemt subsets*
    - *If there were, for example, 5 elements in N (instead of 4), then we would need to define SECs for all 2-element, 3-element, and 4-element subsets of N.*

TSP: Formulation (SECs)

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1,$$
$$\forall S \subset N: 2 \leq |S| \leq |N| - 1$$

- **Example:** $N = \{1, 2, 3, 4\}$
  - Which subsets S to define these constraints?
    - $S \subset N:\ 2 \leq |S| \leq |N| - 1$
    - $|N| = 4$. *So, all 2-node and 3-node subsets must be considered.*
  - The subsets and the corresponding SECs
    - $S = \{1,2\} \Rightarrow x_{13} + x_{14} + x_{23} + x_{24} \geq 1$
    - $S = \{1,3\} \Rightarrow x_{12} + x_{14} + x_{32} + x_{34} \geq 1$
    - *Do the same for* $S = \{1,4\}, S = \{2,3\}, S = \{2,4\}$, *and* $S = \{3,4\}$.
    - $S = \{1,2,3\} \Rightarrow x_{14} + x_{24} + x_{34} \geq 1$
    - $S = \{1,2,4\} \Rightarrow x_{13} + x_{23} + x_{43} \geq 1$
    - *Do the same for* $S = \{1,3,4\}$ *and* $S = \{2,3,4\}$.



Complete LP Formulation

$$\boldsymbol{Min \sum_{i \in N} \sum_{j \in N} d_{ij} x_{ij}}$$

subject to

$$\sum_{j \in N} x_{ij} = 1, \qquad \forall i \in N$$

successor constraint

$$\sum_{j \in N} x_{ji} = 1, \qquad \forall i \in N$$

predecessor constraint

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1, \qquad \forall S \subset N: 2 \leq |S| \leq |N| - 1$$

SECs

$$x_{ij} \in \{0, 1\}, \qquad \forall i, j \in N$$

Classical Vehicle Routing Problem (VRP)

- **Construct a set of delivery routes for vehicles stationed at a central depot which service all nodes and minimizes routing costs**
- **From the TSP to a more general VRP**
  - Construct multiple routes/routes for multiple vehicles
  - Divide stops over routes
  - Find sequence for each route
- **Input**
  - Customers/demand is known and deterministic
  - Vehicle capacity known
  - M vehicles
- **Objective function**
  - Minimize the total travel distance (time/length), and/or routing costs, and/or the number of vehicles needed

VRP (Possible Additional Constraints)

- **Fleet**
  - Size of fleet, homogenous or heterogeneous fleet, vehicle capacity, maximum driving time
- **Nature of demand**
  - Pickups or deliveries, size of loads, due dates, spreading of locations, precedence relations
- **Underlying network**
  - Single or multiple depot(s), directed or undirected arcs, number of arcs on which vehicles can travels
- …

Thank you!

Questions?

Dr. Tri M. Tran
tri.tran@aalto.fi