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Session 9:
Facility location problems
35E00750 Logistics Systems and Analytics

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Discrete and continuous methods

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- **There are two types of facility location methods**
  - Methods for continuous facility location problems seek to find a location (or multiple locations) anywhere in a two-dimensional plane
  - Methods for discrete facility location problems evaluate known (candidate) locations

- **In this course, we consider the following methods**
  - Continuous: Center-of-gravity method
  - Discrete: Factor rating method
  - Discrete: Cost-volume analysis
  - Discrete: Linear programing formulation of facility location problem

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Center-of-gravity method

Center-of-gravity method

- **The center-of-gravity method is a mathematical technique for finding an optimal location for a single facility**
  - Ideal location minimizes weighted (with volume of goods) distance between, for example, warehouse and retailers
- **Method is using**
  - Location of the network
  - Volume of goods to be shipped to those locations
  - (Transportation costs)
- **Objective**
  - Minimize distance/costs (directly proportional to distance and volume)

Center-of-gravity method

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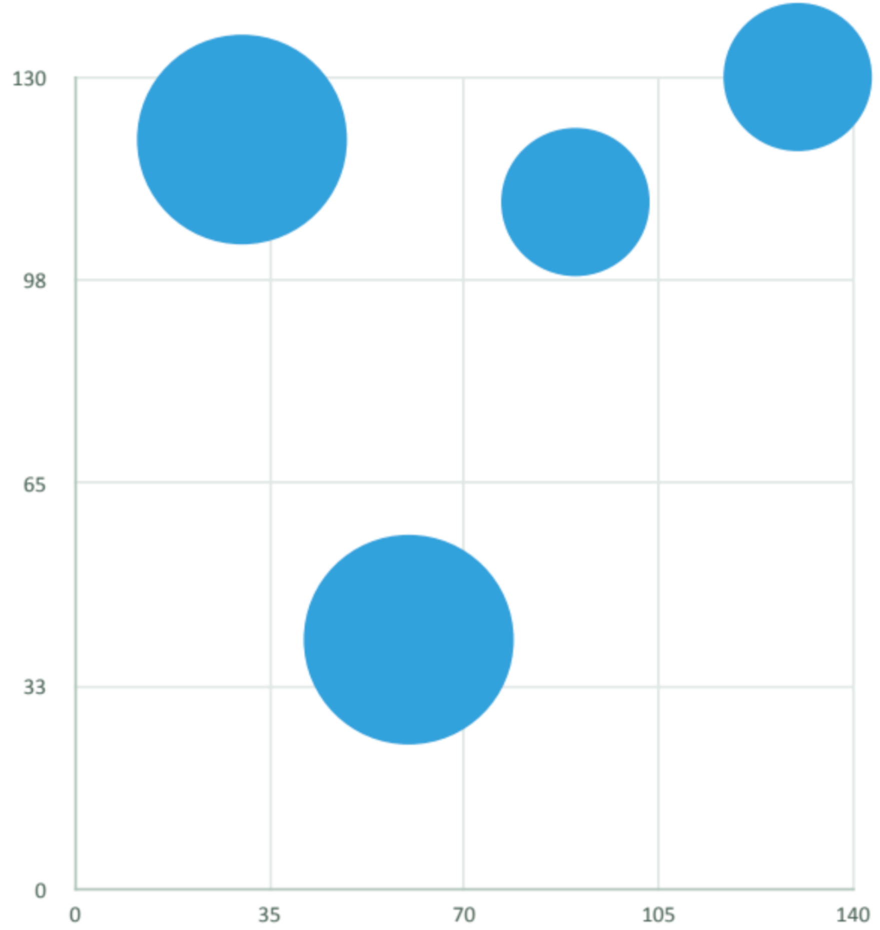
- **Determine an optimal location (x, y) for a warehouse location given the following demand volumes**

City	Demand	Coordinates (x,y)
Amsterdam	2000	(30,120)
Berlin	1000	(90,110)
Osnabruck	1000	(130,130)
Brussels	2000	(60,40)



Center-of-gravity method

C_x = X-coordinate facility location

d_{ix} = X-coordinate of location i

W_i = Volume of goods moved to or from location i

$$C_x = \frac{\sum_i d_{ix} W_i}{\sum_i W_i}$$

C_y = Y-coordinate facility location

d_{iy} = Y-coordinate of location i

$$C_y = \frac{\sum_i d_{iy} W_i}{\sum_i W_i}$$

Center-of-gravity method (example)

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- **Four cities with given coordinates and demand**

City	Demand	Coordinates (x,y)
Amsterdam	2000	(30,120)
Berlin	1000	(90,110)
Osnabruck	1000	(130,130)
Brussels	2000	(60,40)

- **Determine an optimal location (x,y) for a warehouse location**

Center-of-gravity method (example)

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- **Determine an optimal location (x,y) for a warehouse location**

City	Demand	Coordinates (x,y)
Amsterdam	2000	(30,120)
Berlin	1000	(90,110)
Osnabruck	1000	(130,130)
Brussels	2000	(60,40)

- Step 1: Total sum of demand ($\sum_i w_i$)=6000
- Step 2a: Total sum of demand moved to x-coordinate:
  - $\sum_i d_{ix} w_i = 30 * 2000 + 90 * 1000 + 130 * 1000 + 60 * 2000 = 400{,}000$
- Step 2b: Total sum of demand moved to y-coordinate
  - $\sum_i d_{iy} w_i = 120 * 2000 + 110 * 1000 + 130 * 1000 + 40 * 2000 = 560{,}000$

Center-of-gravity method (example)

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- **Determine an optimal location (x,y) for a warehouse location**
  - Step 2 (summary): 400,000 of goods to x-coordinate; 560,000 to y-coordinate
  - Step 3a: x-coordinate warehouse:

$$\frac{\sum_i d_{ix} w_i}{\sum_i w_i} = \frac{400{,}000}{6{,}000} = 66.7$$

  - Step 3b: y-coordinate warehouse:

$$\frac{\sum_i d_{iy} w_i}{\sum_i w_i} = \frac{560{,}000}{6{,}000} = 93.3$$

City	Demand	Coordinates (x,y)
Amsterdam	2000	(30,120)
Berlin	1000	(90,110)
Osnabruck	1000	(130,130)
Brussels	2000	(60,40)

Center-of-gravity method

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- **Optimal location is (66.7, 93.3)**

City	Demand	Coordinates (x,y)
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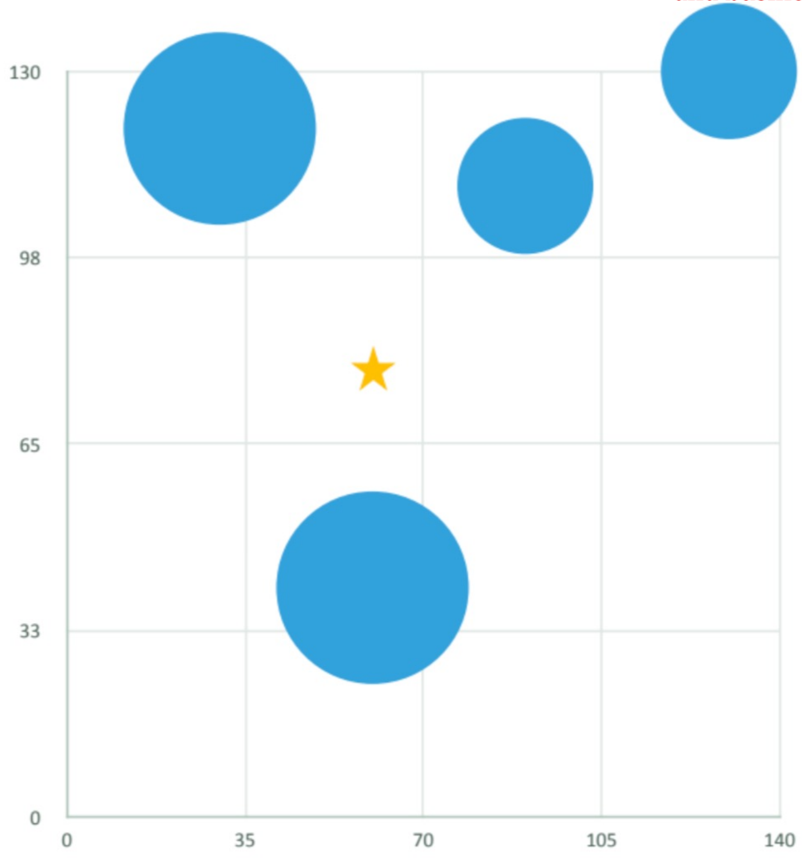
Osnabruck	1000	(130,130)
Brussels	2000	(60,40)



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Factor rating method

Factor rating method

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Factor rating method

- **Deciding between a set of known candidate locations based on qualitative and quantitative factors**
  - List of relevant factors (qualitative)
  - Assign importance weight to each factor (0-1)
  - Develop a scale for each factor (1—100)
  - Score each location using factor scale
  - Multiply scores by weights for each factor and total
  - Select location with maximum total score
- **No exact results due to subjectivity of factors, weights, scales and scores**

Factor rating method

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- **Example inspired from Alibaba warehouse location decision**

Factor	Weight	Score Netherlands	Score Belgium	Weighted score Netherlands	Weighted score Belgium
Total	1.00			50	57

Factor rating method

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- **Example inspired from Alibaba warehouse location decision**

Factor	Weight	Score Netherlands	Score Belgium	Weighted score Netherlands	Weighted score Belgium
Geografical positioning	0.30	60	70	0.30*60=18	21
Possibilities offered by airport authorities	0.25	10	70	0.25*10=2.5	17.5
Tax environment	0.15	80	20	0.15*80=12	3
State of the infrastructure	0.10	85	25	0.10*85=8.5	2.5
Availability of labour	0.20	45	65	0.20*45=9	13
Total	**1.00**			**50**	**57**

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Cost-volume analysis

Cost-volume analysis

- **Analysis to make an economic comparison of a set of known/candidate locations**

- **Determine fixed and variable costs for each known location**
  - Fixed costs (for example, costs of opening a warehouse, or costs of acquiring a truck)
  - Variable costs (for example, cost per product, or cost per kilometer)

Cost-volume analysis (example)

- **Three candidate locations for a warehouse**
  - Assen (fixed cost of €30,000.00 and variable cost of €65.00 per pallet)
  - Heerenveen (fixed cost of €55,000.00 and variable cost of €30.00 per pallet)
  - Groningen (fixed cost of €110,000.00 and variable cost of €10.00 per pallet)

- **Which location would be best at which number of pallets?**

Cost-volume analysis (example)

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- **Let x be the number of pallets**
- **Assen vs. Heerenveen**

$$30000 + 65x = 55000 + 30x$$
$$\Leftrightarrow 35x = 25000$$
$$\Leftrightarrow x = 714.29$$

  - Hence, from 715 pallets, Heerenveen becomes the preferred location
- **Heerenveen vs. Groningen**

$$55000 + 30x = 110000 + 10x$$
$$\Leftrightarrow 20x = 55000$$
$$\Leftrightarrow x = 2750$$

  - Hence, from 2751 pallets, Groningen becomes the preferred location

Cost-volume analysis (example)

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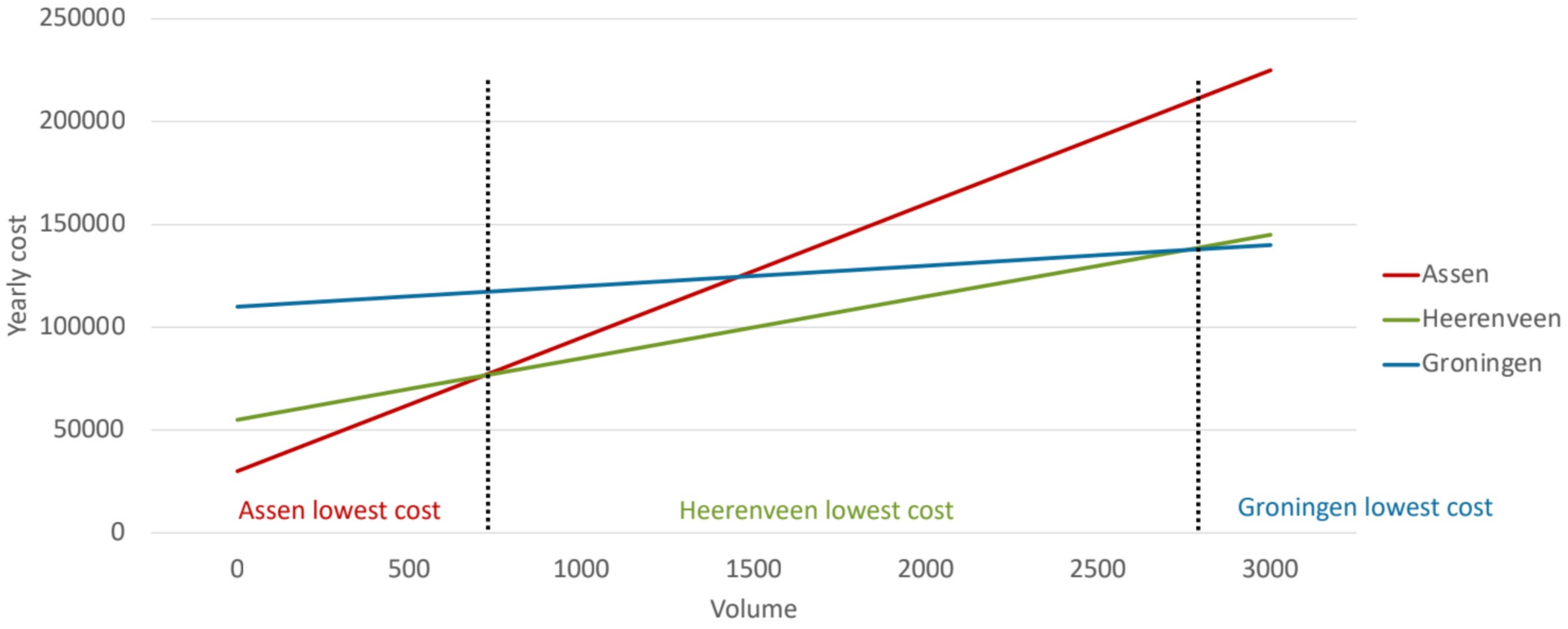

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Linear Programming (LP) Formulation of the Facility Location Problem (FLP)

LP formulation of the FLP

- **General FLP**
  - Set of spatially distributed customers
  - Set of candidate facilities to serve customer demands
  - Distances, time, and costs are measured by a given metric
- **Main questions**
  - Number of facilities
  - Location(s) of facilities
- **Typical objective**
  - Minimize costs (facility costs, transportation costs, inventory costs)

Facility Location Problem

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- **Sets:**
  - I: set of customers
  - J: set of candidate locations
- **Parameters:**
  - D_i: demand amount of customer i
  - K_j: capacity of facility j
  - F_j: fixed cost for opening facility j
  - c_{ij}: cost of sending ***one unit of product*** from location j to customer i
- **Variables:**
  - y_j: whether or not to open a facility at location j
  - x_{ij}: amount of demand of customer i satisfied from location j

Minimize

$$\sum_{j \in J} F_j y_j + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

s.t.

$$\sum_{j \in J} x_{ij} = D_i \qquad \forall i \in I$$

$$\sum_{i \in I} x_{ij} \leq K_j y_j \qquad \forall j \in J$$

$$x_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J$$

$$y_j = 0 \text{ or } 1 \qquad \forall\, j \in J$$

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$$\sum_{i \in I} x_{ij} \le K_j y_j \qquad \forall j \in J$$

$$x_{ij} \ge 0 \qquad \forall i \in I, \forall j \in J$$

$$y_j = 0 \text{ or } 1 \qquad \forall j \in J$$

Minimize the total **fixed (facility opening) costs** and **transportation costs**

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All demands of all customers must be met

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Facility Location Problem

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Minimize

$$\sum_{j\in J} F_j y_j + \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}$$

s.t.

$$\sum_{j\in J} \boldsymbol{x_{ij}} = \boldsymbol{D_i} \qquad \forall \boldsymbol{i} \in \boldsymbol{I}$$

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$$\sum_{i\in I} x_{ij} \le K_j y_j \qquad \forall j \in J$$

$$x_{ij} \ge 0 \qquad \forall i \in I, \forall j \in J$$

$$y_j = 0 \text{ or } 1 \qquad \forall j \in J$$

- If facility j is not opened, no demand can be served from there. (If $y_i = 0$, then $\sum_{i\in I} x_{ij} = 0$).
- If facility j is opened, the maximum amount it can supply is bounded by its capacity. (if $y_i = 1$, then $\sum_{i\in I} x_{ij} \le K_j$)

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Facility Location Problem

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$$y_j = 0 \text{ or } 1 \qquad \forall j \in J$$

Valid range constraints

Facility Location Problem (Uncapacitated)

- **If facilities have infinite (or unrestrictively large) capacities, the constraint**

$$\sum_{i \in I} x_{ij} \leq K_j y_j \quad \forall j \in J$$

  needs to be adjusted.

- **Note that completely removing this constraint is not correct!**
  - We still need to ensure that if facility j is not opened, no demand can be served from there (if $y_j = 0$, then $\sum_{i \in I} x_{ij} = 0$).
- **The following constraint achieves this (with M being a sufficiently large number):**

$$\sum_{i \in I} x_{ij} \leq M y_j \quad \forall j \in J$$

  - If $y_j = 0$, then $\sum_{i \in I} x_{ij} = 0$
  - If $y_j = 1$, then $\sum_{i \in I} x_{ij} \leq M$ (not restrictive, since M is a large number)

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Network Design Problem

Different Network Designs

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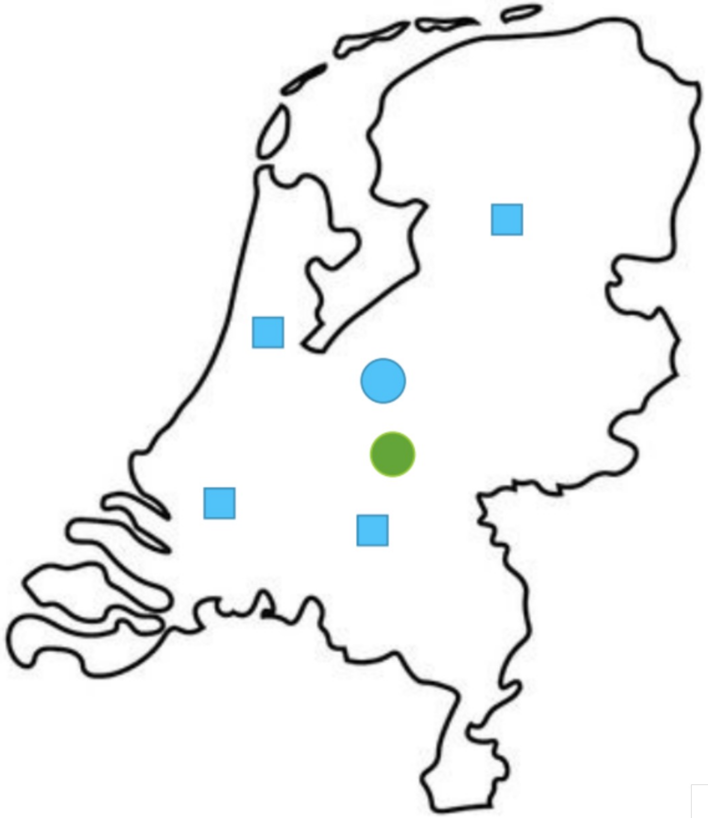
- 1 fresh food distribution center
- 1 non-perishable goods distribution center
- 4 regional distribution centers



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Different Network Designs

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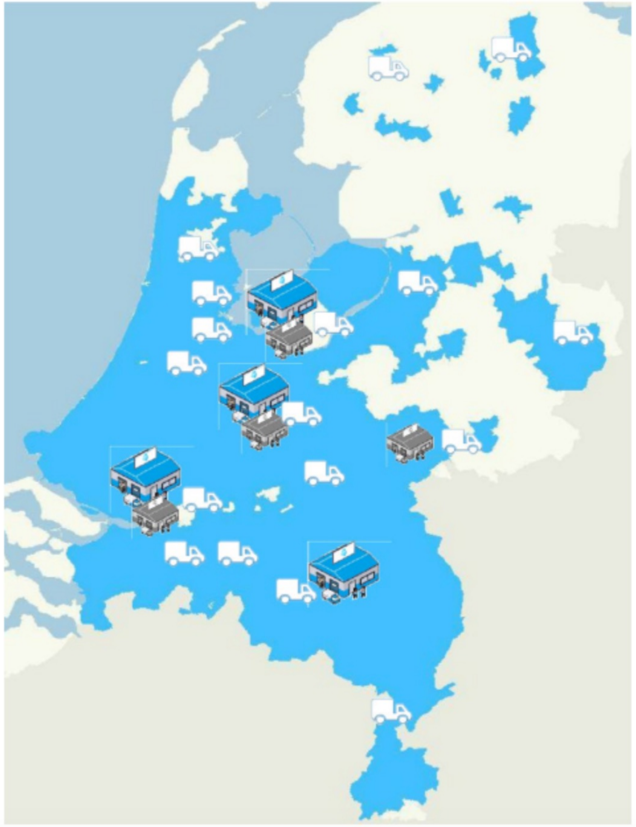
- 4 home shopping centers
- 4 national centers (non-perishable, frozen fresh, bake-off)
- 17 hubs



General Approach to Network Design

- **Given**
  - A set of facilities, and
  - demand/supply quantities of these facilities
- **Find**
  - The routes to be operated
  - The features (frequency, number of intermediate stops, etc.) of the routes to be operated
  - The traffic assignment along the routes
  - The operating rules at each facility
  - Possibly, the relocation of empty vehicles and containers

Network Design Problem

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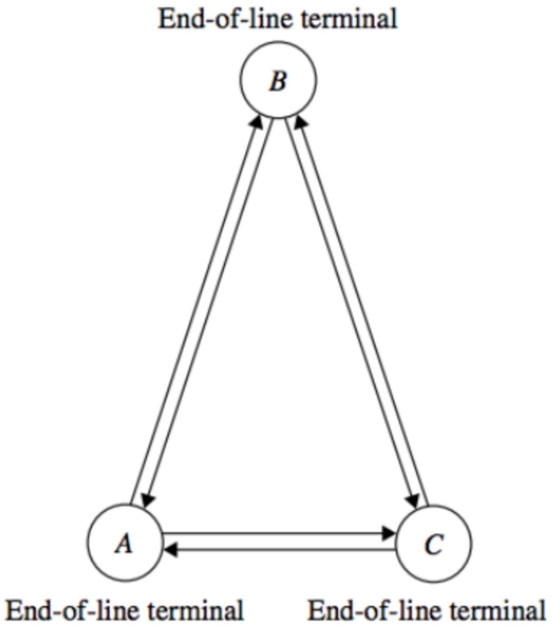
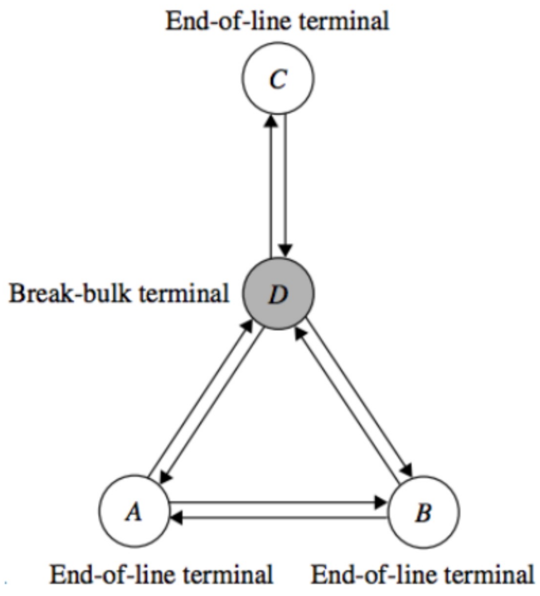
- **Consider two alternative service networks**



Quicker shipments but higher operating cost



Freight consolidation: slower shipments but lower operating cost

Basic Network Design Problem

- **Single commodity**
- **Known demand/supply quantities of each facility (node)**
- **Known costs for each link between facilities (arc)**
  - Fixed cost for opening an arc
  - (Variable cost) transportation cost over an arc
- **Find**
  - Whether or not to open each arc
  - The volume of goods transported on each arc

Step 1: Definition of Parameters

- **Sets**
  - D: Set of demand nodes
  - S: Set of supply nodes
  - N: Set of all nodes ($D \cup S$)
- **Parameters**
  - D_i: demand quantity of node i
  - S_i: supply quantity of node i
  - u_{ij}: capacity of arc (i, j)
  - f_{ij}: fixed cost of opening arc (i, j)
  - c_{ij}: cost of transporting **one unit of product** on arc (i, j)

Step 1: Definition of Variables

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- **Step 1a: What are the variables?**
  - $y_{ij} = \begin{cases} 1, \text{if arc (i, j)is opened} \\ 0, \text{otherwise} \end{cases}$
  - x_{ij}: quantity transported on arc (i, j)

- **Step 1b: Indicate the valid range of all variables**
  - $y_{ij} \in \{0,1\}, \forall i \in N, \forall j \in N$ (binary: y_{ij} values are 0 or 1 for all i and j)
  - $x_{ij} \geq 0,\ \forall i \in N, \forall j \in N$ (x_{ij} non-negative values for all i and j)

Step 2: Define Objective

- **Step 2a: What do you want to achieve?**
  - Minimize total cost, consisting of fixed cost (cost to open a link) and variable costs (per unit transportation costs)

- **Step 2b: Express mathematically**
  - Fixed costs

$$\sum_{i \in N} \sum_{j \in N} f_{ij} y_{ij}$$

  - Variable costs

$$\sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}$$

  - Objective function:

$$Minimize \sum_{i \in N} \sum_{j \in N} f_{ij} y_{ij} + \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}$$

Step 3: Formulating Constraints

- **Balance constraints:**
  - Net flow out of a supply node is equal to the supply quantity
  - Net flow into a demand node is equal to the demand quantity

Total volume of products that go out of node i

$$\sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} S_i & if\ i \in S \\ -D_i & if\ i \in D \end{cases} \qquad \forall i \in N$$

Total volume of products that come into node i

Constraint defined for every node i in the set of nodes N

Step 3: Formulating Constraints

- **Constraints linking the two variables, x_{ij} and y_{ij}**
  - If $y_{ij} = 0$, x_{ij} must be zero as well
  - If $y_{ij} = 1$, x_{ij} can take any value between 0 and the arc capacity (u_{ij})

$$x_{ij} \leq u_{ij} y_{ij}, \qquad \forall i \in N, \forall j \in N$$

Complete Formulation

$$\boldsymbol{minimize} \sum_{i \in N} \sum_{j \in N} f_{ij} y_{ij} + \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}$$

s.t.

$$\sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} S_i \text{ if i} \in \text{S} \\ -D_i \text{ if i} \in \text{D} \end{cases} \quad \forall i \in N$$

$$x_{ij} \leq u_{ij} y_{ij}, \qquad \forall i \in N, \forall j \in N$$

$$y_{ij} \in \{0,1\}, \qquad \forall i \in N, \forall j \in N$$

$$x_{ij} \geq 0, \qquad \forall i \in N, \forall j \in N$$

Basic Network Design Problem

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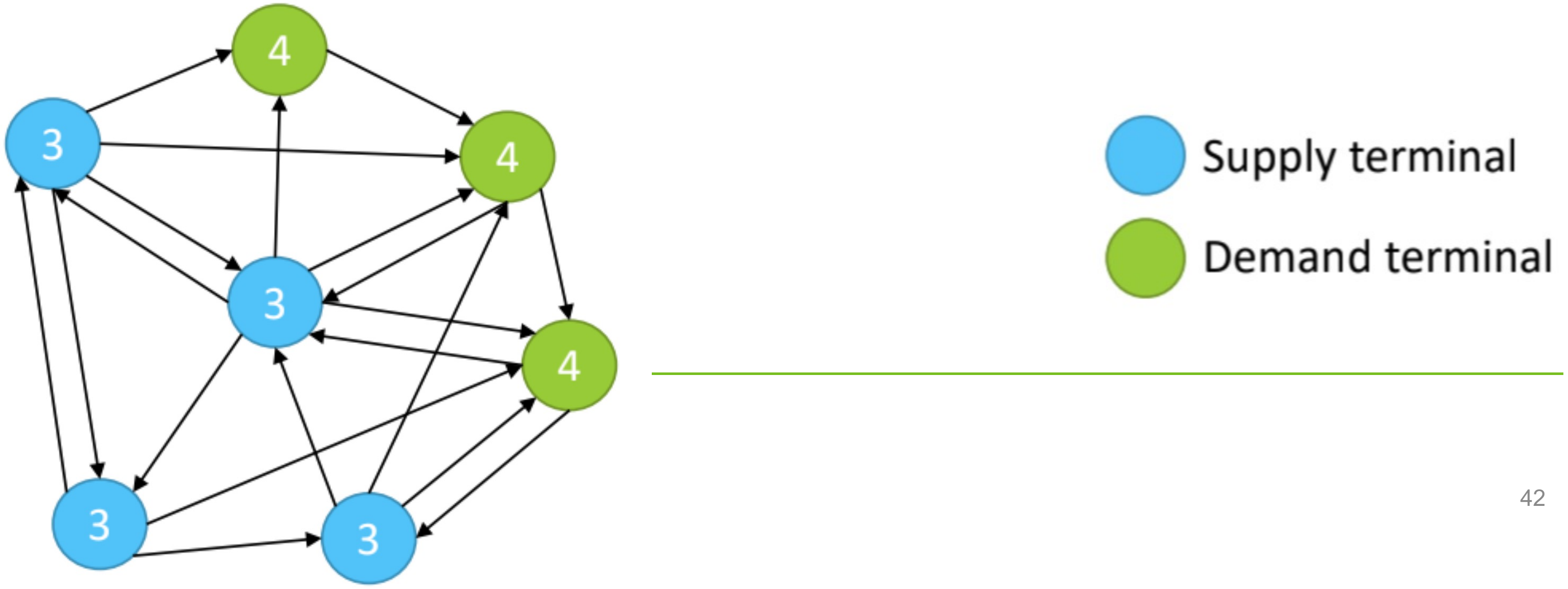
- **Example: 7-node network**
  - Supply nodes: demanding 4 unit loads each
  - Supply nodes: supplying 3 unit loads each
  - There are arcs from every node to every other node (not all of them are drawn on the figure)
  - Which arcs to use? How many unit loads to send on each arc?



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Basic Network Design Problem

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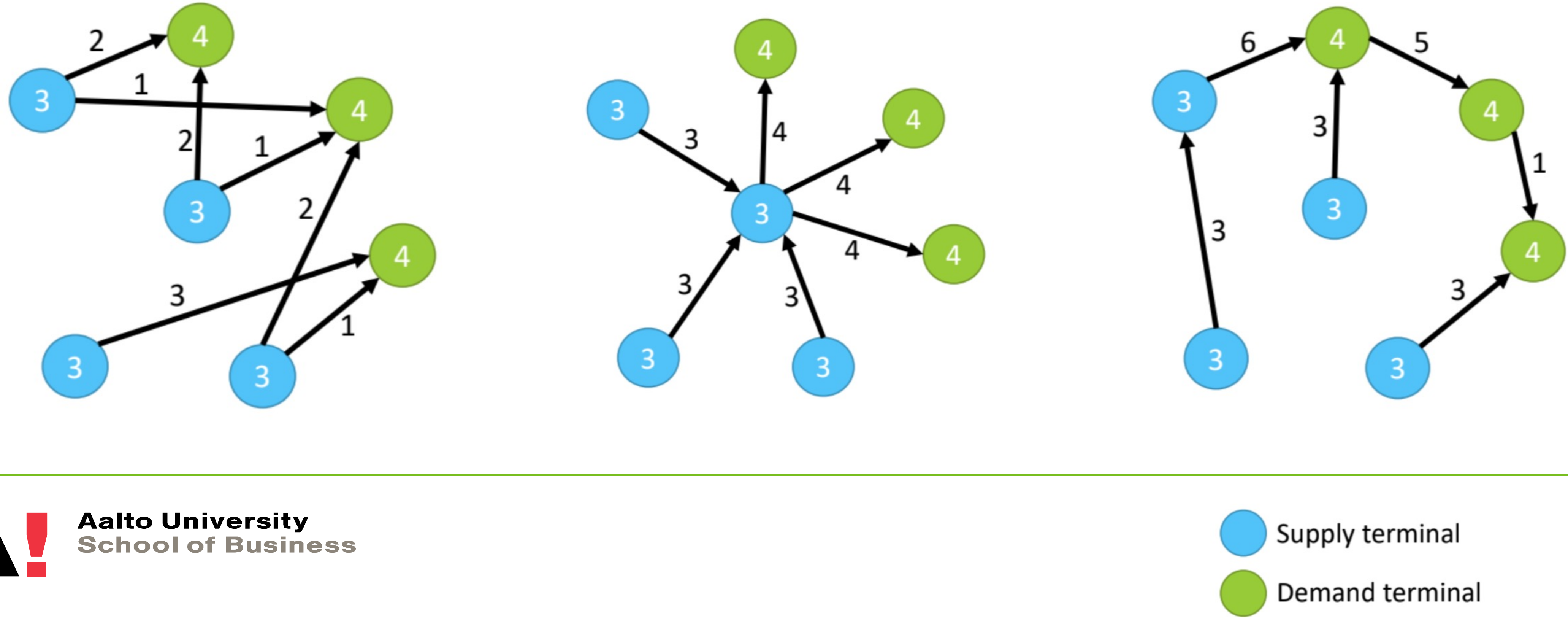
- **Some of the possible solutions**
  - Note that there are many, many more



Network Design Problem (Uncapacitated)

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- **What if some (or all) arcs were uncapacitated)**
  - "Uncapacitated" means infinite capacity
  - Instead of $x_{ij} \leq u_{ij}y_{ij}$, we would get
  - $x_{ij} \leq My_{ij}$

    (where M is a sufficiently large number)

Network Design Problem (Multi-commodity)

- **Multi-commodity network design problem variant**
- **Given demand/supply quantities of each node for each commodity, and costs**
  - Fixed cost for opening an arc
  - (Variable cost) Cost of transporting each commodity over an arc
- **Find**
  - Whether or not to open an arc
  - The volume of **each commodity** transported on each arc

Network Design Problem (Multi-commodity)

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- **Decision variables**
  - $y = \begin{cases} 1, \text{if arc(i, j)is opened} \\ 0, \text{otherwise} \end{cases}$
  - x_{ij}^{k}: quantity of commodity k transported on arc (i, j)
- **Parameters**
  - K: Set of commodities
  - $D(k)$: Set of nodes that demand commodity k
  - $S(k)$: Set of nodes that supply commodity k
  - N: Set of all nodes $(N = U_{k \in K}[S(k) \cup D(k)])$
  - D_i^k: Demand quantity of node i for commodity k
  - S_i^k: Supply quantity of commodity k in node i
  - u_{ij}: Capacity of arc (i, j)
  - f_{ij}: Fixed cost of opening arc (i, j)
  - c_{ij}^k: Unit cost of transporting commodity k on arc (i, j)

Network Design Problem (Multi-commodity)

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- **Complete formulation for multi-commodity capacitated**

$$minimize \sum_{i\in N}\sum_{j\in N} f_{ij}y_{ij} + \sum_{i\in N}\sum_{j\in N}\sum_{k\in K} c_{ij}^k x_{ij}^k$$

s.t.

$$\sum_{j\in N} x_{ij}^k - \sum_{j\in J} x_{ji}^k = \begin{cases} S_i \text{ if } i \in S(k) \\ -D_i \text{ if } i \in D(k) \end{cases}, \qquad \forall i \in N, \forall k \in K$$

$$\sum_{k\in K} x_{ij}^k \le u_{ij}y_{ij}, \qquad \forall i \in N, \forall j \in N$$

$$\sum_{k\in K} x_{ij}^k \le u_{ij}y_{ij}, \qquad \forall i \in N, \forall j \in N$$

$$y_{ij} \in \{0,1\}, \qquad \forall i \in N, \forall j \in N$$

$$x_{ij}^K \ge 0, \qquad \forall i \in N, \forall j \in N, \forall k \in K$$

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Thank you!

Questions?

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