ELEC-E8116 Model-based control systems
/ solutions 4

Problem 1. Let A and B be such matrices that AB and BA are both defined. Prove the so-called “push-through”-rule

$$A(I+BA)^{-1} = (I+AB)^{-1}A$$

Solution.
In this problem, it has been assumed that the inverse matrices exist. In order for both AB and BA to be defined, the dimension of A must be *mxn* and the dimension of B nxm. (Note that even though both matrix products are square matrices, A and B are not necessarily square.)

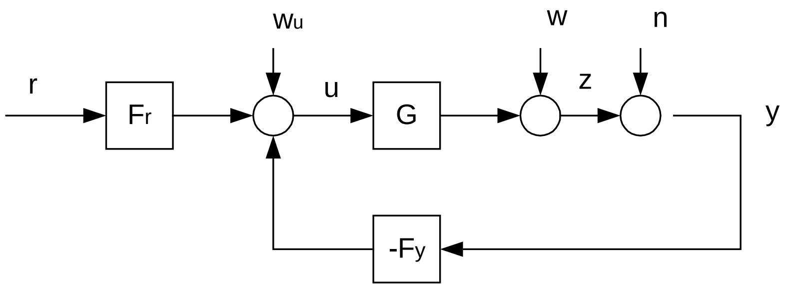
Start from the left side of the equation and multiply by $I + BA$ from the right.

$$A(I+BA)^{-1}(I+BA) = (I+AB)^{-1}A(I+BA)$$
$$\Leftrightarrow$$
$$A = (I+AB)^{-1}(A+ABA) = (I+AB)^{-1}(I+AB)A$$
$$\Leftrightarrow$$
$$A = I * A = A$$

An identity followed. Because we proceeded through equivalences, the claim has been proved.

The term “push-through” is because A is “pushed” from the left in and from the right out.

Problem 2. Consider the closed loop system shown in the figure.



Suppose that the control, load and measurement disturbances are zero. Let there be modeling error in the process transfer function G such that the real process is

$$G_0 = (I+\Delta_G)G$$

where Δ_G is the relative model error.

- **a.** Derive formulas for the real output z_0 and for the output predicted by the model, z.
- **b.** Derive the formulas

$$
\begin{aligned}
z_0 &= (I+\Delta_z)z \\
\Delta_z &= S_0\Delta_G \\
S_0 &= (I+G_0F_y)^{-1}
\end{aligned}
$$

What can be said about the sensitivity of the system?

Solution.
Assuming that the disturbances are zero it follows directly

$$z = Gu = G(F_r r - F_y z) = GF_r r - GF_y z$$

from which

$$z = (I+GF_y)^{-1}GF_r r$$

For the real output correspondingly

$$z_0 = (I+G_0F_y)^{-1}G_0F_r r$$

Try to find the relationship between z and z_0. Considering the model uncertainty leads to

$$
\begin{aligned}
&z_0 + G_0F_y z_0 = G_0F_r r \\
&z + G_0F_y z - \Delta_G GF_y z = GF_r r
\end{aligned}
$$

and by subtracting these from each other

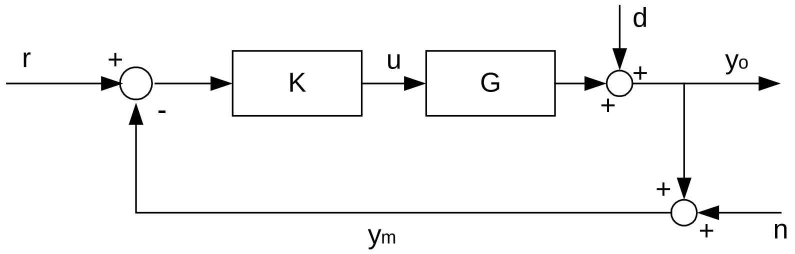
$$z_0 - z + G_0F_y(z_0 - z) = \Delta_G(GF_r r - GF_y z)$$

But $z = GF_r r - GF_y z$, so that

$$z_0 - z = (I+G_0F_y)^{-1}\Delta_G z$$

which gives the formulas of the problem directly. The output sensitivity is small, if the sensitivity function of the real (true) system is small. This property can be added to the result of the next problem also.

Problem 3. Consider the closed loop system shown in the figure.



Write the equations describing the system and identify

a. the closed loop transfer function
b. the sensitivity function
c. the complementary sensitivity function

Write formulas for the output y_0, control u and error $e = r - y_0$. What should be asked from the functions in a-c in order that the system would perform well?

Solution.
By direct calculus

$$y_0 = G(s)u + d$$

$$y_0 = GK(r - y_0 - n) + d \quad \Rightarrow \quad (I + GK)y_0 = GKr + d - GKn$$

$$y_0 = (I + GK)^{-1}GKr + (I + GK)^{-1}d - (I + GK)^{-1}GKn$$

Use the following concepts

$L = GK$	(open) loop transfer function
$G_c = (I + GK)^{-1}GK$	closed loop transfer function
$S = (I + GK)^{-1}$	sensitivity function
$T = (I + GK)^{-1}GK$	complementary sensitivity function

In this configuration $G_c = T$, but this does not hold generally, see textbook, chapter 6. However, generally it holds that

$$S + T = I$$

The output is then

$$y_0 = G_c r + Sd - Tn = Tr + Sd - Tn$$

The error variable is the difference between the reference and the real output variable (not the measured output variable)

$$
\begin{aligned}
e = r - y_0 &= (I - G_c)r - Sd + Tn \\
&= (I - T)r - Sd + Tn \\
&= Sr - Sd + Tn
\end{aligned}
$$

where the facts $G_c = T$ and $S + T = I$ have been used. Correspondingly for the control

$$
\begin{aligned}
u = K(r - y_m) &= K(r - y_0 - n) = KSr - KSd + KTn - Kn \\
&= KSr - KSd + K(T - I)n \\
&= KSr - KSd - KSn
\end{aligned}
$$

The basic frequency responses and diagrams of classical control engineering, the Bode diagram and the Nyquist plot, are plotted for the loop transfer function L (why?).

To get a broader view in the analysis of the system it is useful to use other frequency functions as well.

Demands: The system must be stable (poles of G_c, gain and phase margins calculated from L). The sensitivity function must be small so that the load disturbances are not affecting too much; on the other hand, the complementary sensitivity function must be small because of measurement noise. These demands are in conflict because of $S + T = I$. Also for reference following it should hold that $G_c = T$ should be close to one (identity matrix in MIMO case), which is reasonably big. Thus, compromises must be made in the design. Luckily, the load disturbances usually occur in low frequencies, while the measurement noise is in the high frequency range. The controller should be such that the sensitivity function is small in low frequencies and grows then as the complementary sensitivity function decreases.

There are also other factors (related to robustness) that are important in terms of the sensitivity functions.