ELEC-E8107 - Stochastic models, estimation and control

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Exercises Session 3

Exercise 1

Consider the time-invariant linear Stochastic Differential Equation (SDE), called the Ornstein–Uhlenbeck process:

$$\dot{x}(t) = -\lambda x(t) + \tilde{v}(t) \quad x(0) = x_0 \tag{1}$$

1. Write the ordinary differential equation (ODE) describing the evolution of the mean of the state x.

2. Write the solution of the ODE obtained in the previous question.

The SDE given in the equation (1) cannot be solved using the general numerical methods for deterministic differential equations. This is because the white noise is not differentiable anywhere. Method such as Euler–Maruyama method does not explicitly require continuity. Such method can be used to solve the SDE in equation (1). Using Euler-Maruyama method, the equation (1) can be written in discrete form as:

$$x(k+1) = x(k) - \lambda x(k)\Delta t + \Delta v(k) \tag{2}$$

Where $\Delta v(k) \sim \mathcal{N}(0, Q\Delta t)$; Q is the spectral density of the white noise $\tilde{v}(t)$. Assuming $\lambda = \frac{1}{2}, \quad \Delta t = \frac{1}{100}, \quad x(0) = 1, \quad Q = 1$:

3. Using equation (2), simulate 500 trajectories of the Ornstein–Uhlenbeck process on the time interval $t \in \begin{bmatrix} 0 & 1 \end{bmatrix}$. (Show the result on a plot)

4. Check that the mean trajectory approximately agree with the theoretical value (use a plot to show the results).

Exercise 2

A vehicle is moving at near constant speed following a line (1D-motion). The position is measured with a measurement covariance of $r = 0.5$. A kinematic model for such a system is as follow:

$$\begin{aligned} p(k+1) &= p(k) + \Delta t v(k) + \frac{\Delta t^2}{2} w(k) \\ v(k+1) &= v(k) + \Delta t w(k) \end{aligned} \tag{3}$$

where $w(k) \sim \mathcal{N}(0, q)$. The position and speed, at each instant k, can be combined in a single vector $x(k)$ considered as the state of the system.

1. Write the equation (3) in a state-space form: $x(k+1) = Fx(k) + Lw(k)$. Specify the variables F, and L.

2. Given that the state of interest is composed of the position and the speed, and that only the position is measured with white noise, write the measurement equation for the system.

3. Implement, using MATLAB, a Kalman filter to estimate the state $x(k)$ of the system at each time instant.

Note: For the implementation of the Kalman filter, you are given two files containing the real data (*RealData.mat*) and the measurement data (*measurement.mat*); You don't need to modify these files and they are already loaded in the Matlab script (*BasicKalmanFilter.m*) in which you need to implement the Kalman filter. Some basic instructions are given in the Matlab script to help you write the Kalman filter.