Problem 5.1: Newton's Method for a Quadratic Problem

Consider the following quadratic optimization problem

$$\min_{x} \; f(x) = x^\top A x \tag{1}$$

with variable vector $x \in \mathbb{R}^n$, and where $A \in \mathbb{R}^{n \times n}$ is a positive semidefinite (PSD) matrix.

(a) Not all PSD matrices are symmetric. Show that if the PSD matrix A in (1) is not symmetric, we can always replace it with a symmetric PSD matrix B such that $x^\top A x = x^\top B x$.

(b) Show that Newton's method converges in one iteration when applied to the problem (1).

(c) Show that Newton's method converges in one iteration when applied to the following quadratic problem with an additional linear term:

$$\min_{x} \; f(x) = x^\top A x - b^\top x \tag{2}$$

with variables $x \in \mathbb{R}^n$. Assume that $A \in \mathbb{R}^{n \times n}$ is a symmetric PSD matrix and $b \in \mathbb{R}^n$.

Problem 5.2: Affine Invariance of Newton's Method

Consider the following unconstrained optimization problem

$$\min_{x} \; f(x) \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function. Show that Newton's method applied to problem (3) is *affine invariant*, meaning that the progress of Newton's method is independent of affine transformations of the original problem (e.g., scaling, translation, and rotation).