ELEC-E8116 Model-based control systems
/exercises and solutions 5

Problem 1. Consider the process

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$$

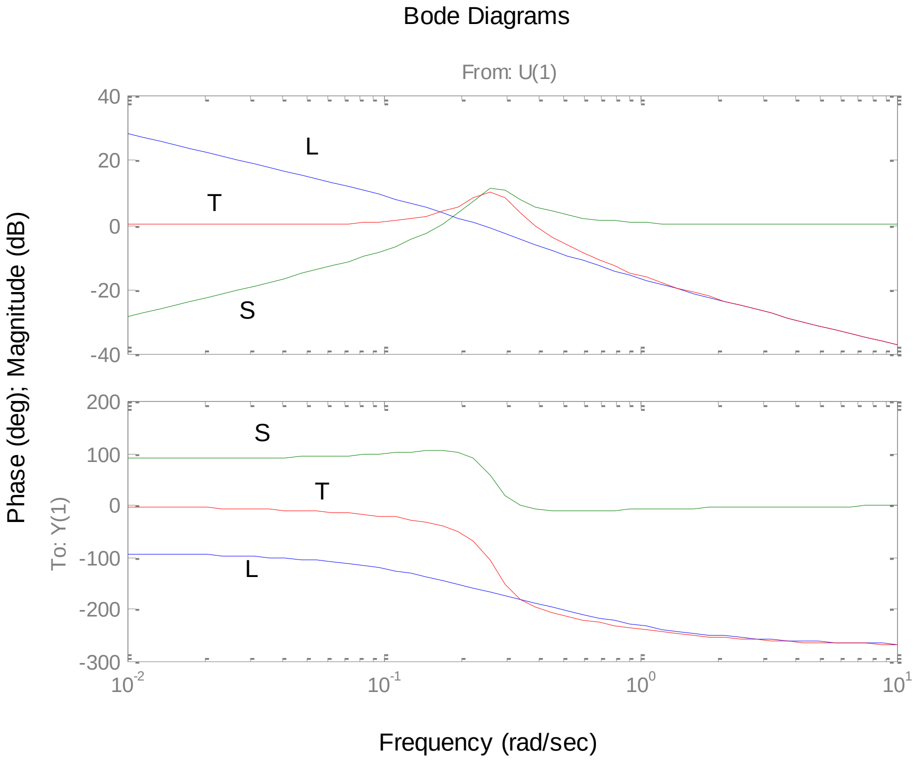
which is controlled by the *PI*-controller

$$K(s) = 1.136(1 + \frac{1}{12.7s})$$

Plot the *L*-, *S*- and *T*-curves. Determine the gain crossover frequency, gain and phase margins and bandwidth.

Solution.
In this control configuration $F_y = F_r = K$ (figure 6.1 in the textbook). We obtain

$$L = GK \qquad S = (1+L)^{-1} \qquad T = (1+L)^{-1}L = 1 - S$$



Matlab:

```
G=3*tf([-2 1],conv([5 1],[10 1]));

L=G*K;
S=1/(1+L);
T=1-S;
w=logspace(-2,1);
bode(L,S,T,w)
```

Gain and phase margins

```
[Gm,Pm,Wcg,Wcp] = MARGIN(L);
```

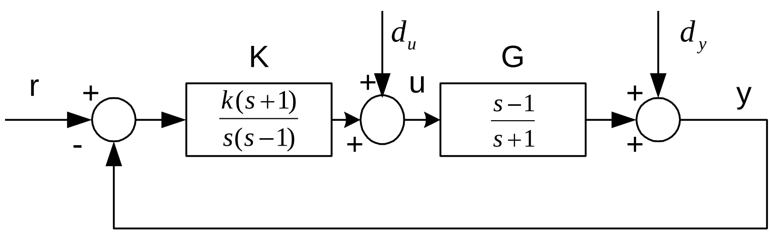
Gain margin is 1.64, phase margin 19.4 degrees, gain crossover frequency $\omega_c = 0.236$.

Bandwidth can be defined in several ways; heuristically it means the frequency range, in which control is efficient. Sometimes the gain crossover frequency is used as the bandwidth, sometimes it is the frequency, at which the sensitivity function *S* first time crosses $1/\sqrt{2} \approx 0.707$ (-3 dB) from below . Perhaps the most common definition is that frequency, for which the complementary sensitivity function *T* crosses first time that value –3dB from above. By Matlab

```
[mag,phase]=bode(T,w);
```

it can be approximated $\omega_B \approx 0.45$.

Problem 2. Consider the control configuration in the figure, in which the parameter *k* is positive. Is the system internally stable?



Solution.
The disturbance signals are d_u and d_y (reference *r* can be set to the value zero; through the reference no unstable mode can appear, which would not be detected in what follows). Internal stability can be checked by calculating the transfer functions from the disturbance signals to the outputs *y* and *u*.

$$u = (1+KG)^{-1}d_u - K(1+GK)^{-1}d_y$$
$$y = G(1+KG)^{-1}d_u + (1+GK)^{-1}d_y$$

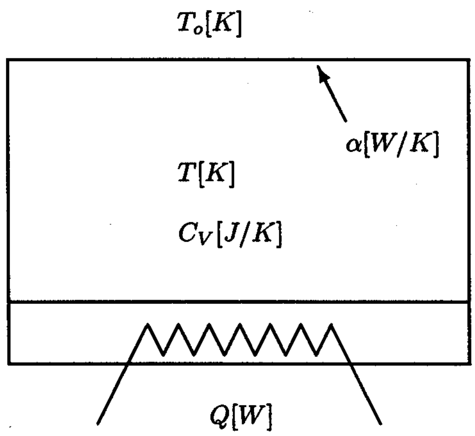
(the above functions can be written in slightly different forms, note the ”push-through”-rule)

It is immediately seen that

$$G_{d_y u}(s) = -K(1+GK)^{-1} = -\frac{k(s+1)}{(s-1)(s+k)}$$

which is unstable so that the system is internally unstable.

Problem 3. Consider the apartment heating system shown in the figure. The aim is to control the incoming heat such that the room temperature is as desired with the accuracy of one Kelvin. The outdoor temperature is a disturbance.



a. Derive a model of the system and linearize it.
b. Scale the model in such a way that the transfer functions related to the process and disturbance are comparable with each other.

Solution.
Let us write the energy balance equation expressing that the change of heat energy in the room is equal to the difference of incoming and outflowing energy (per time unit).

$$\frac{d}{dt}(C_V T) = Q + \alpha(T_0 - T)$$

in which T [K] is the room temperature, C_V [J/K] is the room heat capacity, Q [W] is the inflowing heat energy and the term $\alpha(T_0 - T)$ [W] describes the outflowing energy (ventilation and heat flow through the walls). Consider the case $Q^* = 2000$ W and T^*-$T_0^* = 20$ K. From the stationary solution of the heat equation we obtain $\alpha^* = 2000/20$ W/K = 100 W/K. Further, let $C_V = 100$ kJ/K.

Assuming that α is constant, it is clear that the energy balance equation is linear. Consider small changes with respect to the operating point

$$\delta T(t) = T(t) - T^*(t),\ \delta Q(t) = Q(t) - Q^*(t),\ \delta T_0(t) = T_0(t) - T_0^*(t)$$

which gives

$$C_V \frac{d}{dt}\delta T(t) = \delta Q(t) + \alpha\left[\delta T_0(t) - \delta T(t)\right]$$

By taking the Laplace transformation and assuming that $\delta T(t) = 0$ when $t = 0$ it follows

$$\delta T(s) = \frac{1}{\tau s + 1}\left[\frac{1}{\alpha}\delta Q(s) + \delta T_0(s)\right]; \quad \tau = \frac{C_V}{\alpha}$$

The time constant is $\tau = 100 \cdot 10^3$ /100s = 1000 s ≈ 17 min., which seems realistic. It means that after a step change in the incoming heat flow it takes 17 minutes before the room temperature has increased 63% of the total change.

Concerning the scaling of the variables see the textbook sections 5.2 and 7.1. Use the following scaled variables

$$y(s) = \frac{\delta T(s)}{\delta T_{\max}}; \quad u(s) = \frac{\delta Q(s)}{\delta Q_{\max}}; \quad d(s) = \frac{\delta T_0(s)}{\delta T_{0,\max}}$$

When it is desired that the room temperature varies plus/minus one kelvin from the operating point, take

$$\delta T_{\max} = \delta e_{\max} = 1\,K$$

Correspondingly, if the input heat energy can vary 2000 W and the outdoor temperature 10 K, we obtain

$$\delta Q_{\max} = 2000\,W; \qquad \delta T_{0,\max} = 10\,K$$

The scaled process model is of the form

$$y(s) = G(s)u(s) + G_d(s)d(s)$$

where

$$G(s) = \frac{1}{\tau s + 1}\frac{\delta Q_{\max}}{\delta T_{\max}}\frac{1}{\alpha} = \frac{20}{1000s + 1}$$

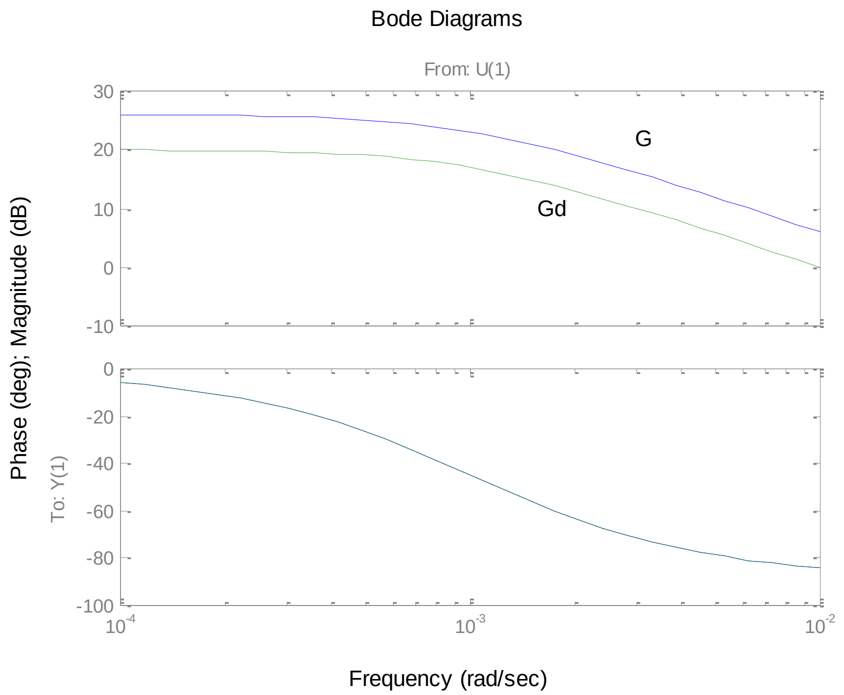
$$G_d(s) = \frac{1}{\tau s + 1} \frac{\delta T_{0,\max}}{\delta T_{\max}} = \frac{10}{1000s + 1}$$

From the Bode diagram it is seen that a full compensation of the disturbance is possible, because

$$|G(i\omega)| > |G_d(i\omega)|, \quad \forall \omega$$

(see theorem 7.7 in the textbook)

This means that by using controls $|u| < 1$ the maximal disturbances



$|d| = 1$ can be compensated (e = 0).

It is important to notice the meaning of scaled variables. Transfer functions related to different physical variables can be made comparable such that the absolute values of signals in the controlled system are less than one in magnitude, at least most of the time.