31E11100 - Microeconomics: Pricing
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Hints for Problem Set 1

1. An individual consumer has quasi-linear preferences with utility of consumption $v(q)$ given by

$$v(q) = \begin{cases} q - \frac{q^2}{2} \text{ for } 0 \leq q \leq 1 \\ \frac{1}{2} \text{ for } q > 1 \end{cases} .$$

   (a) Derive the individual demand function $q(p)$ of this consumer.

   Solution.

   The utility of the consumer i is given by:

$$u_i = v_i(q_i) + y_i$$

   The consumer faces a budget constraint:

$$m_i = pq_i + y_i$$

   The consumer maximizes her utility subject to the budget constraint:

$$\max_{q_i} u_i(q_i)$$
$$s.t.\ m_i = pq_i + y_i$$

   Subsitute constraint in and take FOC with respect to q_i and solve for q_i

$$1 - q_i - p = 0$$
$$q_i = 1 - p$$

(b) Derive the optimal linear pricing strategy of a monopolist who faces such a consumer, and who has cost function

$$c(q) = cq,$$

where $0 < c < 1$.

Solution. Firm problem:

$$\max_{q_i} p * q - cq$$
$$s.t.\ q = 1 - p$$

Substitute constraint in and take FOC with respect to q and solve for q

$$1 - 2q - c = 0$$
$$q = \frac{1-c}{2}$$

Solve for price using the demand equation:

$$p = \frac{1+c}{2}$$

(c) Suppose the monopolist can use a two-part tariff, i.e. a fixed fee plus a linear price component. Derive the optimal two-part tariff.

Solution.

Two-part tariff consist a fixed fee which I denote by f and a linear price which I denote by p. The monopolist must set the fixed fee such that the consumers decides to buy. Consumers buy if they receive a positive consumer surplus given the linear price. This "participation constraint" can be written:

$$CS(p) \geq f$$

With linear demand consumer surplus is simply a triangle (think about Micro 1) and is given by:

$$CS(p) = \frac{(1-p)^2}{2} = \frac{q^2}{2}$$

In the optimum the monopolist sets the fixed fee equal to consumer surplus. The optimization problem is then:

$$\max_{q_i} f + p * q - cq$$
$$s.t.\ q = 1 - p$$
$$\frac{q^2}{2} = f$$

Substitute both constraints in and take FOC with respect to q:

$$q + 1 - 2q - c = 0$$
$$q = 1 - c$$

Solve for linear price and fixed component:

$$p = c$$
$$f = \frac{(1-c)^2}{2}$$

2. A buyer has a unit demand and valuation $v = 1$ for a product. There is a large number of sellers in the market, and it is assumed that the prices offered by the sellers are independently distributed according to a uniform distribution in $[0, 1]$. However, to get a price quote from a seller, the buyer must incur a search cost $c > 0$ (per price quote). Given multiple price quotes, the buyer chooses the lowest price.

   (a) Suppose that the buyer chooses to get only 1 price quote. What is the expected price that she pays, and what is her expected total payoff? What is the expected payoff if the buyer chooses to get 2 quotes? What if she asks n quotes, where n is just some number?

   Solution.

   If you draw n values from $Uniform[0, 1]$ distribution, the expectation of the lowest value (the so-called first-order statistic) is

$\frac{1}{1+n}$.[1]

If n=1 the payoff is given by:

$$v - E[p|n = 1] - c = \frac{1}{2} - c$$

If n=2 the payoff is given by:

$$v - E[p|n = 2] - 2c = \frac{2}{3} - 2c$$

With n searches we have

$$v - E[p|n] - nc = 1 - \frac{1}{1+n} - nc$$

(b) Formulate this "fixed sample search" problem of the buyer, i.e. write the optimization problem of the buyer who chooses n to maximize her expected payoff. How do you expect the optimal value of n to vary in c? If you can, solve the problem explicitly.

Solution.

$$\underset{n}{max}\; v - [nc + \frac{1}{1+n}]$$

Take FOC with respect to n and solve for it:

$$-c + (1+n)^{-2} = 0$$
$$n = \frac{1}{\sqrt{c}} - 1$$

[1]This is because for uniformly distributed random variables in $[0,1]$, the kth order statistic of n draws is distributed as $X_{(k)} \sim Beta(k, n+1-k)$. Therefore $X_{(1)} \sim Beta(1, n)$, and the expectation of such distribution is $\frac{1}{1+n}$. We will not consider the specific properties of order statistics in this course.

Lower c will increase search in equilibrium.

(c) Suppose now that the buyer searches sequentially. Conside first the simplest case, i.e. suppose that the buyer has already received one price quote, say p, and has a chance to ask for one more quote at cost c. For which values of p should the buyer ask for another quote?

Solution.

The buyer will continue searching if:

$$\begin{aligned} U(\textit{buy in next period}) &\geq \; U(\textit{buy now}) \\ 1 - [c - \mathbf{E}(min\{p_1, p_2\})] &\geq \; 1 - p_1 \\ c &\leq \; p_1 - \mathbf{E}(min\{p_1, p_2\} \end{aligned}$$

The RHS is zero when $p_1 = 0$, and larger than c when $p_1 = 1$ (otherwise the buyer will not search even once). Because RHS is also non-decreasing in p_1,[2] there exists a threshold $\bar{p} \in (0, 1)$ such that the buyer gets the second quote as long as $p_1 > \bar{p}$.

If you want to solve the problem explicitly, note that

$$\begin{aligned} \mathbb{E}(min\{p_1, p_2\}) &= \Pr(p_1 \leq p_2)\mathbb{E}(min\{p_1, p\}|p_1 \leq p_2) + \Pr(p_1 > p_2)\mathbb{E}(min\{p_1, p_2\}|p_2 < p_1) \\ &= [1 - F(p_1)]p_1 + F(p_1)\mathbb{E}(p_2|p_2 < p_1) \end{aligned}$$

Here $F(x) = x$ (and $f(x) = 1$ for $x \in [0, 1]$) given our distributional assumption for prices. Moreover, we have

$$\mathbb{E}(p_2|p_2 < p_1) = \frac{\int_0^{p_1} zf(z)dz}{\Pr(p < l)} = \frac{\int_0^{p_1} zdz}{F(p_1)} = \frac{\int_0^{p_1} zdz}{F(p_1)} = \frac{p_1^2}{2p_1} = \frac{p_1}{2}$$

Thus we get:

$$\mathbb{E}(min\{p_1, p_2\}) = [1 - F(p_1)]p_1 + F(p_1)\mathbb{E}(p_2|p_2 < p_1) = (1 - p_1)p_1 + p_1(\frac{p_1}{2}) = p_1 - \frac{p_1^2}{2}$$

[2] A mathematical result on minima and expectations states that $\mathbb{E}(min\{p_1, p_2\}) \leq min\{\mathbb{E}(p_1), \mathbb{E}(p)\} = min\{p_1, \frac{1}{2}\} \leq p_1$.

And the condition for continuing search becomes:

$$c \leq \frac{p_1^2}{2} \iff \sqrt{2c} \leq p_1$$

That is, the buyer searches if the price quoted at period 1 is at least as low as $p_1 = \sqrt{2c}$.

(d) Suppose that the buyer searches sequentially as long as she wishes, i.e. she asks for one quote at the time, and decides after each quote whether to buy at the lowest offer so far or whether to continue asking for another quote. Formulate the problem of the buyer, and discuss the nature of the solution. How do you expect the solution to vary in c. Again, if you can, solve the problem explicitly.

Solution.

Nothing much changes compared to the previous case! Key here is that it does not matter how many times the buyer has searched before; those are sunk costs. If the new quoted price is above the lowest price quoted prior to the new price the problem of the buyer stays the same.

(e) Is the buyer better off with fixed sample or sequential search (i.e. one in b. or in d.)? Why? How would you modify the model to make this question more interesting?

Solution.

With the assumptions made in this questions, sequential search is always preferred as it allows the buyer to make the same choice as with fixed sample search (choosing n quotes) but the buyer may also stop earlier if she got lucky. Thus, in a sense, she has "more options".

To introduce a tradeoff between sequential and fixed sample search, we could assume that searching sequentially is more costly (e.g. it takes more time which needs to be discounted appropriately).

Then the buyer would need to balance between searching for multiple quotes now or once each period until the low-enough-price is found.

(f) Discuss economic situations where one or the other form of search (fixed sample/sequential) might be more appropriate.

Solution.

A natural example of sequential search is e.g. grocery shopping. Fixed sample search may then describe situations such as selling a firm, any kind of competitive tendering, or buying e.g. a renovation, electricity, flights and insurance for yourself.

3. A municipality wants to procure a service. There are $N = 2$ identical firms, who decide simultaneously whether or not to make a price quote for the service. Preparing the price quote costs $c > 0$, but there are no production costs. The municipality has a reservation value $R > 0$ for the service. The firm that offers the lowest price p gets a deal and makes profit $p - c$ (as long as $p \leq R$; in case there is a tie with m firms offering the lowest price, each of them makes profit $p/m - c$). Those who offer a price that is not the lowest, get no deal and make profit $-c$, and those firms that do not offer anything get 0.

(a) Formalize this as a simultaneous move game between the firms. What is the set of strategies available to each firm?

Solution.

- A set of players (firms) $i \in \mathcal{I} = \{1, 2\}$
- A set of strategies, $s_i \in S_i = \mathbb{R}^+ \cup N$ for all $i \in \mathcal{I}$. Choosing N means *not offering a price at all*, choosing some $s_i \in \mathbb{R}^+$ means offering that price.

- Payoffs $u_i(s_1, s_2) = \begin{cases} s_i - c & \text{if } s_i < s_j \\ \frac{s_i}{2} - c & \text{if } s_i = s_j \\ -c & \text{if chosen } s_i > s_j \\ 0 & \text{if chosen } s_i = N \end{cases}$

A game is then formally a collection strategies and payoffs for both players, (S_1, S_2, u_1, u_2).

(b) Assume first that $c = 0$. Find a Nash equilibrium of the game.

Solution.

Nash equilibrium is $p_1 = p_2 = 0$.

I first show that $p_1 = p_2$ by contradiction. Consider $p_i > p_j$. If $p_j = 0$ then $u_j = 0$ while choosing p_i would yield $u_i = p_i/2 > 0$. Given this choosing p_i is a profitable deviation for firm j. Therefore for $p_i > p_j$ to be an equilibrium it must be that $p_j > 0$. But if this is the case then firm i has a profitable deviation. It can cut its price to p_j to earn $p_j/2$.

Next I show that it must be that $p_1 = p_2 = 0$ Consider that $p_1 = p_2 > 0$, then both firms wold find to deviate just a little bit because: $(p_i - \epsilon) > p_i/2$, when ϵ is close to zero.

It cannot be a nash equilibrium that either one firm makes an offer or that neither makes an offer. In both cases there is a profitable deviation. One can check this using similar arguments as above.

(c) Assume that $c > 0$. Does the game have a Nash equilibrium in pure strategies?

Solution.

To show that there cannot be *any* pure strategy NEs, let's start by thinking how many sellers post a price. Neither of the firms offering a price cannot be a NE. If the other one does not offer a price then the other seller could benefit by posting a price: $c < p < R$. Moreover, if only one seller offers some price s_i, she would like to offer R given the other sellers strategy (there is no competition). However, if $s_i > c$, the other seller would like to offer $s_i - \varepsilon$ for some small $\varepsilon > 0$ to get a positive payoff instead of not offering anything and getting zero. Thus, only one seller offering a price cannot be a NE. Finally, suppose that both would offer some price. Then, regardless of prices offered, the seller offering the highest price would always like to either undercut the seller with the lower price (if the lowest price is higher than c), or not offer a price at all (if the lowest price is equal to or lower than c). So sellers both offering a price cannot be a NE either. Therefore, we do not have any NE in *pure strategies*.

(d) Derive a symmetric mixed strategy equilibrium with an atomless price distribution with support $[c, R]$. What is the expected profit of each firm in this equilibrium? What is the probability of an individual firm making a positive price offer? What is the equilibrium price distribution?

Solution.

I start by showing that in symmetric equilibrium both must make zero profits.

In symmetric equilibrium, both sellers play the same strategy and therefore have the same expected profit. We argue first that this expected profit cannot be negative, and then show that it cannot be positive either. Thus it must be zero.

Suppose that both sellers would make negative expected profit in equilibrium. Then each seller could benefit by deviating to play pure strategy $s_i = N$ and get zero profit. Thus such situation cannot be a NE.

Suppose then that both sellers would make positive expected profit in equilibrium. Because both sellers are *mixing* between two or more different strategies, they must be indifferent between playing those as pure strategies (otherwise they would not mix). First, we know that choosing $s_i = N$ gives zero profit, hence all sellers must participate and offer a price. Second, suppose they both mix between offering some *fixed amount* of different prices (i.e. their strategies are discrete), then it is always possible to end up in a tie. This cannot be the case as both sellers would like to undercut and thus win the sale. Therefore, if both sellers participate, they must mix between offering some continuous distribution of prices. But if they do so, offering the highest price in the support will have negative expected profit (such offer *never* wins). Therefore, in a NE, the expected profit cannot be positive either.

Consequently, any symmetric NE with positive or negative expected profit is not possible and thus in such NE, expected profit must be zero.

Using this zero profit condition we can derive the probability of a firm making a positive price offer. Denote the probability of making an offer by π. Conditional on making an offer, the cumulative density function of prices is denoted by $F(p)$. Firms must be indifferent between offering any price from the support of $F(p)$ and not offering anything (otherwise they would not play mixed strategies). The one offering R and winning with probability $(1-\pi)^{n-1}$ (=no other offers) must be indifferent between offering that and c (and winning for sure). Writing this down gives

$$
\begin{aligned}
Pr\{win|p\}p - c &= 0 \\
[Pr(\text{offer below } p_j) + Pr(\text{no offer from j})]p - c &= 0 \\
[(1 - F(p)) + (1 - \pi)]p - c &= 0 \\
(1 - \pi)R - c &= 0 \\
\pi &= 1 - \frac{c}{R}
\end{aligned}
$$

Thus no offers will be made if $c = R$, and neither will make an offer if $c = 0$.

(e) Does this model have any other symmetric equilibria?

Solution.

Yes. There is no pure strategy equilibrium as shown above. Also there cannot be an equilibrium with an atom in the distribution. With an atom in the distribution the probability of a tie is positive and in a tie both sellers would find it optimal to cut their prices just a little bit.

(f) (bonus: harder) Can you generalize your answer in d) to any $N \geq 2$? What is the probability distribution of the number of firms that make a price offer for a fixed N? What is the limiting probability distribution of the number of firms that make an offer in the limit where $N \to \infty$?

Solution.

With $N > 2$ firms we have that:

$$\pi = 1 - \sqrt[n-1]{c/R}$$

Taking the limit of this yields the limiting probability of firms making an offer. A useful thing to notice is that the number of firms making an offer is binomially distributed. Once we take the number of firms to infinity the distribution converges to a Poisson distribution.

4. **Third-degree price discrimination.** A pharmaceutical company sells a given drug in two geographically separated markets, denoted A and B. The demands are given by $Q_A(p_A) = 1 - p_A$ and $Q_B(p) = \frac{1}{2} - p_B$. For simplicity, the transport and production costs are assumed to be zero.

   (a) Suppose that the firm sets a uniform price across the two markets. What is the profit-maximizing uniform price, and what are the quantities sold at that price in the two markets?

   Solution. With uniform prices we have $p_A = p_B \equiv p$. The monopolist then solves

$$\max_p \quad pQ_A(p) + pQ_B(p)$$
$$s.t. \quad Q_A(p) = 1 - p, Q_B(p) = \frac{1}{2} - p$$

   Note that $pQ_A(p) + pQ_B(p) = p(1-p) + p(\frac{1}{2} - p) = \frac{3}{2}p - 2p^2$. By taking the FOC, we find the profit maximizing uniform price $p^* = \frac{3}{8}$. Thus quantities demanded are $Q_A(\frac{3}{8}) = \frac{5}{8}$ and $Q_B(\frac{3}{8}) = \frac{1}{8}$.

   (b) Suppose that the firm can set different prices in the two markets. What are the profit-maximizing prices and what are the quantities sold in the two markets?

   Solution. Now the firm just solves the usual monopolist problem separately for both markets. The profit maximizing prices are $p_A^* = \frac{1}{2}$ and $p_B^* = \frac{1}{4}$, and the respective quantities sold are $Q_A(\frac{1}{2}) = \frac{1}{2}$ and $Q_B(\frac{1}{4}) = \frac{1}{4}$. Thus, compared to the uniform price case, less products are sold in market A while more products are sold in market B.

   (c) Compute the producer's and consumers' surpluses under a uniform price and under geographical price discrimination. Compare the two situations and discuss.

Solution. We compute the surplus of consumers in both markets and the producer surplus of the monopolist in both the uniform and the separate pricing cases. Under the optimal uniform price p^*, these are

$$CS_A^u = \frac{Q_A(p^*)(1-p^*)}{2} = \frac{25}{128}$$
$$CS_B^u = \frac{Q_B(p^*)(\frac{1}{2}-p^*)}{2} = \frac{1}{128}$$
$$PS^u = Q(p^*)p^* = (Q_A(p^*) + Q_B(p^*))p^* = \frac{30}{128} + \frac{6}{128} = \frac{36}{128}$$

Under the optimal separate prices for both markets, p_A^* and p_B^*, the surpluses are

$$CS_A^s = \frac{Q_A(p_A^*)(1-p_A^*)}{2} = \left(\frac{1}{2}\right)^2/2 = \frac{16}{128}$$
$$CS_B^s = \frac{Q_B(p_B^*)(\frac{1}{2}-p_B^*)}{2} = \left(\frac{1}{4}\right)^2/2 = \frac{4}{128}$$
$$PS^s = Q_A(p_A^*)p_A^* + Q_B(p_B^*)p_B^* = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{32}{128} + \frac{8}{128} = \frac{40}{128}$$

Total surplus higher with uniform pricing.

(d) Do the insights of c) hold generally? What if the demand in market B is changed to $Q_B(p) = \frac{1}{3} - p_B$?

Solution. No. If $Q_B(p) = \frac{1}{3} - p_B$, we find for uniform pricing that $p^* = \frac{1}{2}$ (note that this is superior to $p^* = \frac{1}{3}$ as both prices imply no sales in market B), which implies quantities demanded $Q_A(\frac{1}{2}) = \frac{1}{2}$ and $Q_B(\frac{1}{2}) = 0$. Thus there will be no sales in market B. The surpluses are then:

$$CS_A^u = \frac{Q_A(\frac{1}{2})(1-\frac{1}{2})}{2} = \frac{1}{8} = CS_A^s$$
$$CS_B^u = 0$$
$$PS^u = Q(\frac{1}{2})\frac{1}{2} = (Q_A(\frac{1}{2}) + Q_B(\frac{1}{2}))\frac{1}{2} = \frac{1}{4}$$

For separate pricing we find $p_B^* = \frac{1}{6}$. While the surpluses and pricing for market A are as in part c), we now have $CS_B^s = \frac{Q_B(\frac{1}{6})(\frac{1}{3}-\frac{1}{6})}{2} = \left(\frac{1}{6}\right)^2/2 = \frac{1}{72}$ and the producer surplus extracted from market B is $Q_B(p_B^*)p_B^* = \frac{1}{36}$, thus the total producer surplus is $PS^s = \frac{32}{128} + \frac{1}{36} = \frac{10}{36} > \frac{1}{4} = PS^u$.

Both the producer and consumers in market B get higher surplus under separate pricing, while consumers in market A are indifferent (and do not hence benefit from uniform pricing). Now the total surplus is higher under separate pricing.