31E11100 - Microeconomics: Pricing
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Hints for Problem Set 2

1. A firm sells a product in a market where there are two types of consumers, $\theta \in \{\theta^H, \theta^L\}$. Assume that the mass of consumers is normalized to one, and there are equally many of both types of consumers. All consumers have a unit demand, and θ denotes the reservation value of consumer type θ. Let $\theta^H = 12$ and $\theta^L = 5$. The marginal cost of production is $c = 1$.

   (a) Find the profit maximizing price for the firm.

   Solution.
   Payoffs of the consumers (agents) are $u_i(\theta, q, t) = \theta^i v(q) - t^i$ for both $i = H, L$.[1] The profit of the firm (payoff of the principal) is simply $u_P(\theta, q, t) = t - c(q)$.

   Setting $t < 5$ is not optimal as everybody would buy the product also for $t = 5$. Setting $t \in (5, 12)$ is not optimal either as high types would by also for $t = 12$. Neither type wil not buy the product for $t > 12$. This would yield zero profit and is not optimal. The firm will only sell to the high types by setting $t = 12$ for profit $u_P = 12/2 - 1/2 = 5.5$. This is higher than the profit $u_P = 5 - 1 = 4$ for selling to everyone at $t = 5$.

   (b) The firm then considers producing an additional, lower quality version of the good (a "damaged good"). The damaged version of the good can be produced at constant marginal cost $c = 1.5$, and the reservation value of that good is 4 for the high type, and 3 for the low type. Find the optimal prices for the two versions of the product for the firm that wants to maximize its profit.

   Solution.
   Given this setup, the problem of the firm is to maximize its profit (payoff) by designing a menu of prices for both qualities. In the maximization, the firm must ensure that the menu is both *incentive compatible* ie. high type chooses (q^H, t^H) and low type chooses (q^L, t^L), and that both types want to *participate* meaning that *individual rationality* constraint is satisfied.

[1] Note that here $v(q) = 1$ if the consumer buys the good, and $v(q) = 0$ is she doesn't.

The problem of the firm is given by:

$$\max_{(q^L,t^L),(q^H,t^H)} \lambda(t^H - c(q^H)) + (1-\lambda)(t^L - c(q^L))$$

$$s.t. \qquad \theta^H - t^H \geq \theta^H_D - t^L \qquad (IC^H)$$

$$\theta^L_D - t^L \geq \theta^L - t^H \qquad (IC^L)$$

$$\theta^H - t^H \geq 0 \qquad (IR^H)$$

$$\theta^L_D - t^L \geq 0, \qquad (IR^L)$$

where $\lambda = 1/2$, $c(q^H) = 1$ and $c(q^L) = 1.5$. From lecture notes we know that in these sort of problems the IC for the high type must bind and that the IR for the low type must bind. These two constrains imply the other constraints are satisfied and we can write the problem followingly:

$$\max_{(q^L,t^L),(q^H,t^H)} \lambda(t^H - c(q^H)) + (1-\lambda)(t^L - c(q^L))$$

$$s.t. \qquad \theta^H - t^H = \theta^H_D - t^L \qquad (IC^H)$$

$$\theta^L_D - t^L = 0, \qquad (IR^L)$$

We do not need to take the maximum of this problem because we can simply solve for the transfers from the two constraints. We get that $t^H = 11$ and $t^L = 3$.

(c) Should the firm introduce the damaged version of the good? Why? What are the welfare implications of introducing the damaged version?

Solution.
In part a) the profits of the firm were 5.5. In part b) they are: $\frac{1}{2}11 + \frac{1}{2}3 - 0.5 - 0.75 = 5.75$. Introducing the damaged good is thus optimal for the firm. In part a) the good was sold only to high types and they got zero consumer surplus. In part b) the good is sold to both types. The low types make no consumer surplus but the high types makes a consumer surplus of 1. Everyone is better off!

2. Amazon.com has a single cover price for the books that it sells, but it has a menu of different delivery options ranging from 1-2 days to two weeks. Let's have a model of second-degree price discrimination to explain this.

(a) Assume that buyers have different valuations for fast delivery. This is captured by a parameter θ, where value $\theta = \theta^H$ captures consumers with high valuation for fast delivery and $\theta = \theta^L$ captures consumers with low valuation for fast delivery. Let s denote the actual delivery time and assume the following payoff function:

$$v(\theta, s) = \begin{cases} \theta(1-s), & \text{if } s \leq 1 \\ 0 \text{ otherwise.} \end{cases}$$

Interpret this payoff function (i.e. invent a story that rationalizes the function).

Solution.

Suppose you have an exam date at $s = 1$ and need a book to study. Students who study more get a better grade and thus higher payoff. High type students spend more time studying per day than low types and thus benefit more from getting the book earlier. If the book arrives after the exam, it has no value to students. Note that throughout my solution I assume that consumers value faster delivery and thus $\theta_i > 0$.

(b) Let the cost of delivery at time s be $c(s) = 2(1-s)^2$ for $0 \leq s \leq 1$ and $c(s) = 0$ for $s > 1$. Does this function make sense?

Solution. Yes. The sooner the delivery, the more consumers need to pay; the cost of delivery is decreasing in delivery time s because $c'(s) = -4(1-s) \leq 0$ for $s \in [0,1]$. If the delivery is after the exam, nobody is willing to pay for it.

(c) What are the first best delivery times for the two types of buyers? I.e. how would the seller choose $\widehat{s}^H$ and $\widehat{s}^L$ for θ^H and θ^L respectively if she could see the type of the buyer? What would the corresponding prices $\widehat{t}^H$ and $\widehat{t}^L$ be for those delivery times?

Solution. First-best levels are found by equating marginal cost and marginal utility for both types:

$$c'(s) = -4(1-s) = -\theta = \frac{\partial v(\theta, s)}{\partial s}$$
$$\implies \widehat{s}^H = 1 - \frac{\theta^H}{4} \quad \text{and} \quad \widehat{s}^L = 1 - \frac{\theta^L}{4}$$

The transfers are found by plugging this into the utility function. Thus:

$$\widehat{t}^H = \theta^H(1 - s^H_{FB}) = \frac{(\theta^H)^2}{4}$$
$$\widehat{t}^L = \theta^L(1 - s^L_{FB}) = \frac{(\theta^L)^2}{4}$$

(d) Would the menu $\{(\widehat{s}^H, \widehat{t}^H), (\widehat{s}^L, \widehat{t}^L)\}$ be incentive compatible if the seller does not see θ?

Solution.

No, the menu is not incentive compatible. The first-best offer derived in part c) leaves high type with no surplus but by mimicking the low type, she would get:

$$\theta^H(1 - \widehat{s}^L) - \widehat{t}^L = (\theta^H - \theta^L)\frac{\theta^L}{4} > 0$$

Mimicking is possible because the type of the consumer is private information!

(e) Suppose that fraction λ of the buyers are of type θ^H and $(1-\lambda)$ are of type θ^L. Solve for the profit maximizing incentive compatible menu of delivery times and prices for the seller. For what parameter values should the seller offer two different delivery times?

Solution. Again we know that for the low type the IR constraint binds and for the high type the IC constraint binds.

The IR constraint for the low type is:

$$\theta^L(1-\widehat{s}^L) - \widehat{t}^L = 0$$
$$t^L = \theta^L(1-s^L)$$

The IC constraint for the high types is:

$$\theta^H(1-\widehat{s}^H) - \widehat{t}^H = \theta^H(1-\widehat{s}^L) - \widehat{t}^L$$
$$t^H = \theta^H(1-s^H) - (\theta^H - \theta^L)(1-s^L)$$

The problem is then to maximize profit subject to binding IC^H and IR^L by choosing optimal delivery times s^H and s^L, that is:

$$\max_{s^L,s^H} \lambda(t^H - c(s^H)) + (1-\lambda)(t^L - c(s^L))$$
$$s.t.\ t^H = \theta^H(1-s^H) - (\theta^H - \theta^L)(1-s^L)$$
$$t^L = \theta^L(1-s^L)$$

Plug in the constraints, take the derivative with respect to delivery times. From this you get that:

$$s^H_{SB} = 1 - \frac{\theta^H}{4}$$
$$s^L_{SB} = 1 + \frac{\lambda\theta^H - \theta^L}{4(1-\lambda)}$$

The allocation of the high type is not distorted. The book is delivered to the low type before the exam ($s \leq 1$) only if:

$$\lambda \leq \frac{\theta^L}{\theta^H}$$

You get this inequality by checking when the second term in RHS of the optimal delivery time for low type is negative. Finally, the optimal transfers can be calculated by inserting optimal delivery times s^H_{SB}, s^L_{SB} into the expressions for the transfers above (see IC for high type and IR for low type).

3. A monopolist sells two products $i \in \{1, 2\}$. There is a unit mass (continuum) of consumers, who each have independent valuations for the two goods. Assume that the valuations are uniformly distributed over the unit interval, i.e. $v_i \sim U[0,1]$, $i = 1, 2$. The production cost is assumed to be zero for the seller.

   (a) Suppose the monopolist sells the two products only separately, i.e. sets separate prices p_1 and p_2 for the two products, and lets each consumer decide which product(s) to buy. Compute optimal separate prices and the corresponding profit.

Solution. For the a) part the key is simply to derive the demand curves from the valuations of the consumer, which are uniformly distributed between 0 and 1. We already did this last time. The demand for both products individually are given by:

$$q_i = 1 - p_i$$

The maximization problem is simply:

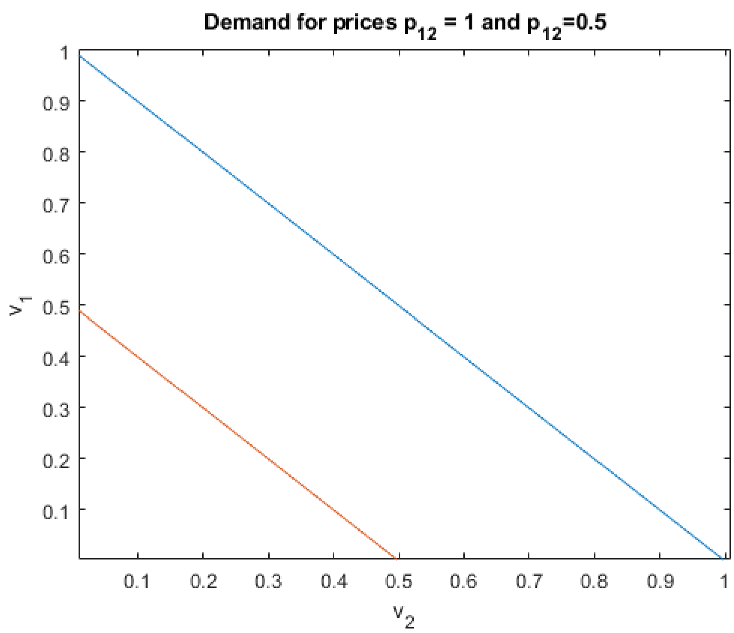
$$\max\ p_1 q_1 + p_2 q_2$$
$$s.t.\ q_1 = 1 - p_1$$
$$q_2 = 1 - p_2$$

Plug in constraints and take partial derivatives with respect to prices. This yields $p_1 = p_2 = \frac{1}{2}$.

(b) Suppose next that the monopolist sells the two products as a bundle only (pure bundling). What is the demand function for the bundle, i.e. the total quantity bought at a given bundle price p_b? To derive that demand, it is helpful to draw a unit square with axes v_1 and v_2 that represents the set of possible player types. For a given bundle price p_b, what is the region in that figure representing those consumers that buy? The demand is then just the area of that region. What would be the profit of the seller if she chooses bundle price $p_b = p_1 + p_2$, where p_1 and p_2 are the ones you derived in a)? Would buyers be better or worse off?

Solution.

The key here is to derive what is the demand (as a function of the price) for the bundle. As suggested in the problem great way to think about this is to draw a unit square with the axis representing the valuations for the two products. This is illustrated in the figure below



From the figure we see that the demand is simply the whole region minus the triangle in the bottom-left region of the graph. Thus we have that $q_{12} = (1 - \frac{1}{2}{p_{12}}^2)$.

If the bundle price is above 1 then demand is the triangle in the up-right corner: $q_{12} = \frac{(1-(p_{12}-1))^2}{2}$.

Bundle price would be one if we would use the individual prices derived above. We are asked to evaluate whether consumer are on average better off with bundle pricing. In general the consumers who have a high valuation for both good prefer bundling while those who have a high valuation for one good and low valuation for the other prefer seperate selling.

Formally the consumer surplus with seperate pricing is given by:

$$CS_{sep} = \int_{p_1}^{1} (v_1 - p_1) dv_1 + \int_{p_2}^{1} (v_2 - p_2) dv_2 = \frac{1}{4}$$

With bundling the consumer surplus is given by:

$$\begin{aligned} CS_{bun} = \int_{0}^{p_{12}} [\int_{p_{12}-v_2}^{1} (v_1 + v_2 - p_{12}) dv_1] dv_2 + \int_{p_{12}}^{1} [\int_{0}^{1} (v_1 + v_2 - p_{12}) dv_1] dv_2 \\ = 1 - p_{12} + \frac{1}{6}{p_{12}}^3 = \frac{1}{6} \end{aligned}$$

where the first part represents consumers who have valuation: $v_2 < p_{12}$ and $v_1 > p_{12} - v_1$ and the second part consumers with valuation: $v_2 > p_{12}$ and $1 > v_1 > 0$.

Consumers would be worse off if the price of the bundle would be one but the welfare effects might be different when the price of the bundle is set optimally.

(c) What is the optimal bundle price p_b and the associated profit for the seller? (in the case where the good is sold only as a bundle)
Solution.
First we need to determine whether the optimal bundle price will be above or below 1. With bundling the demand becomes more elastic and this implies that the bundle price must be below 1. This can be shown by considering how a small price increase by ϵ would affect demand in the case of selling the products seperately and in the case of selling the bundle.

Given the demand derived above for the case were the bundle price is below one the maximization problem is:

$$\max\ p_{12}q_{12}$$
$$s.t.\ q_{12} = (1 - \frac{1}{2}{p_{12}}^2)$$

Plugging in the constraints and taking partial derivative with respect to the bundle price yields: $p_{12} = \sqrt{\frac{2}{3}}$.

(d) Finally, consider mixed bundling, where the seller offers prices $(\overline{p}_1, \overline{p}_2, \overline{p}_b)$, i.e. $\overline{p}_1$ the price of good 1 separately, $\overline{p}_1$ the price of good 2 separately, and $\overline{p}_b$ the price of the bundle containing both goods. Derive the demands for products 1 and 2 and for the bundle for the given $(\overline{p}_1, \overline{p}_2, \overline{p}_b)$. Again, utilize the graphical representation suggested in (b).

Solution.

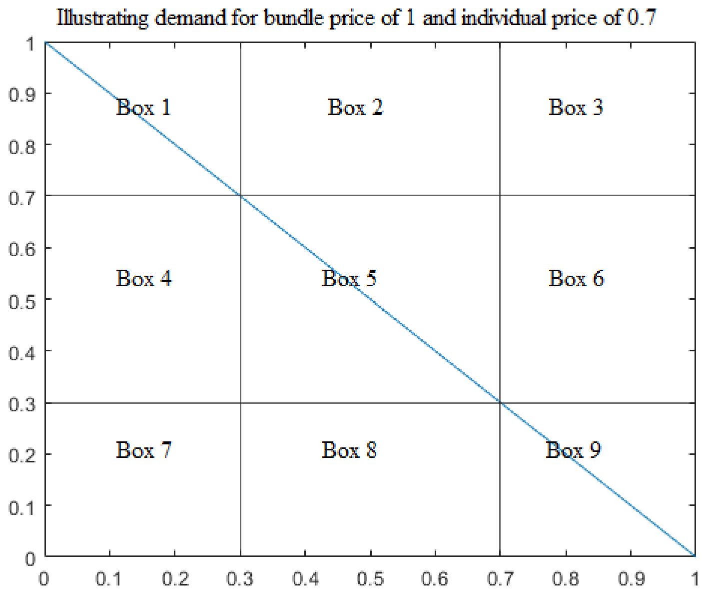
Here I start deriving the demand by considering the consumer who is indifferent between buying the bundle and buying a single product. The indifferent consumer must satisfy:

$$v_i - p_i = v_1 + v_2 - p_{12}$$

From this we can derive:

$$\widehat{v_1} = p_{12} - p_2$$
$$\widehat{v_2} = p_{12} - p_1$$

From now on I denote both individual prices just by p. Adding these two lines and also the individual price lines to the graph in previous page we get:



Now the graph is split into nine different regions that I denote "boxes". Demand for individual products are given by Box 1 and 9. The area in these two boxes is simply $q_1 = q_2 = (p_{12} - p)(1 - p)$. The demand for the bundle is simply the sum of box 2, 3, 6 and half of box 5. We get that $q_{12} = (1 - p_{12} + p)^2 - \frac{1}{2}(2p - p_{12})^2$.

(e) You can take it as given that the seller will in such a case always choose $\overline{p}_1 = \overline{p}_2 := \overline{p}_s$. What would be the optimal prices $(\overline{p}_s, \overline{p}_b)$ in the case of mixed bundling?

Solution.

The problem of the firm is then to maximize the following the expression:

$$max\ 2p[(p_{12} - p)(1 - p)] + p_{12}[(1 - p_{12} + p)^2 - \frac{1}{2}(2p - p_{12})^2]$$

Taking the partial derivative with respect to the bunde price and the individual price yields: $p = \frac{2}{3}$ and $p_{12} = \frac{1}{3}(4 - \sqrt{2})$.

(f) Which mode of pricing (separate, pure bundling, mixed bundling) is best for the seller?

Solution.

Mixed bundling is the best for the firm. To show this calculate profits in parts a), c) and e) given the optimal prices.

4. A monopoly firm operates in a market with a sesonally varying demand. The inverse demand function in season $i \in \{s, w\}$ is given by:

$$p^i = \alpha^i - \beta^i q^i,$$

where $\alpha^s = 10$, $\alpha^w = 8$, $\beta^s = 1$, $\beta^w = 2$. There is a constant marginal cost of production so that producing q^i units in season i costs cq^i, where $c = 1$.

(a) Assume first that there is no capacity constraint for the firm. What is the optimal production level and correponding price for each season?

Solution.

Let's consider the optimal price for summer firts. The optimization problem is (I have already plugged in the inverse demand):

$$max\ q^s(10 - q^s) - q^s$$

Taking derivative with respect to q_s yields $q_s = 4.5$ and $p_s = 5.5$. Going through the same steps with winter we get that $q_w = \frac{7}{4}$ and $q_w = \frac{18}{4}$

(b) Keep on assuming that there is no capacity constraint for the firm. If the firm must use the same price in both seasons, what is the total quantity sold over the two seasons for given price p? What is the optimal price p?

Solution.

Now the optimization probelm of the firm is:

$$max\ pq^s + pq^w - q^w - q^s$$
$$s.t.\ q_s = 10 - p$$
$$q_w = 4 - \frac{p}{2}$$

The optimal price is $p = \frac{31}{6}$.

(c) Assume next that there is a fixed capacity level k. The capacity sets the maximum output that the firm can produce in either season: $q^i \leq k$ for $i \in \{s, w\}$. What is the optimal production level and correponding price for each season as a function of k?

Solution.

Now the optimization probelm of the firm is:

$$\begin{aligned} max\ pq^s + pq^w - q^w - q^s \\ s.t.\ p_s = 10 - q_s \\ p_w = 8 - 2 * q_w \\ q_s \leq k \\ q_w \leq k \end{aligned}$$

Setting up the Lagrangian:

$$L = q_s(10 - p_s) + q_w(8 - 2q_1) - q_w - q_s + \lambda_1(k - q_s) + \lambda_2(k - q_w)$$

Taking derivative with respect to q_s and λ_i yields:

$$\begin{aligned} q_s &= \frac{9 + \lambda_1}{2} \\ q_w &= \frac{7 + \lambda_2}{4} \\ q_s &\leq k \\ q_w &\leq k \end{aligned}$$

If the capacity constraints do not bind then $\lambda_i = 0$ and the solution is as in part a). If the capacity constraints binds then the firm produces the maximum amount and the prices are $p_s = 10 - k$ and $p_w = 8 - 2k$.

(d) Assume now that the firm must choose its production capacity to serve the market, and it costs fk to build k units of capacity. The firm maximizes its total profits over the two seasons, and can choose different prices in the two seasons. For what values of f is the capacity constraint binding in the high season only? Solve the optimal capacity and supply levels for the two seasons both in the case where the capacity constraint binds only in one season and where it binds in both seasons.

Solution.

If capacity constraint binds only in the high season we have $q_s = k$ and $p_s = 10 - k$. The firms problem in the high season is $max\ (10 - k)k - k - fk$. We get that $k = \frac{9-f}{2}$. For this solution to be valid it must be that the demand in the low-season is lower than the optimal k derived before. This holds when:

$$q_w \leq k = \frac{9-f}{2}$$
$$\frac{9}{4} \leq k = \frac{9-f}{2}$$
$$\frac{9}{2} \geq f$$

If this holds for the low season the firms problem in the low season is $max\ (8-2q_w)q_w - q_w$. We get that $q_w = \frac{7}{4}$. Note the cost of capacity can ignored in the low season optimization problem because this cost is allocated fully to the high season which generates the marginal revenue from added capacity. The prices for both seasons can be simply read from the corresponding demand curves.

If the capacity constraint binds in both seasons then we have $q_w = q_s = k$. The optimization problem is then the following:

$$max\ k(10-k) + k(8-2k) - k - k - fk$$

Maximizing this with respect to k yields $k = \frac{16-f}{6}$. Prices can again be read off from the demand curve. The price for the high season will be higher than for the low season.