31E11100 - Microeconomics: Pricing
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Hints for problem set 3

1. We consider here pricing of an experience good. There is a unit mass of consumers with unit demand and reservation valuation uniformly distributed between 1 and 5, i.e. $v \sim U[1,5]$. However, being an experience good, the consumers do not know their individual valuations before consuming (so, before consuming, each individual just knows that her value is drawn from $v \sim U[1,5]$). After consuming once, a consumer learns perfectly her reservation valuation. There are two periods of consumption, and no discounting between the periods. The production cost is zero for the firm.

   (a) Let us consider the second period first. What is the optimal price that the monopolist charges in the second period if no consumer consumed in the first period (i.e. all consumers are still uninformed about their valuations)?

   Solution.

   I denote price in period t by p_t. Because no consumer purchased the good in the first period no one knows their valuation in period 2. The expected valuation is then given by:

$$\mathbb{E}[v] = \int_1^5 vf(v)dv = \int_1^5 \frac{v}{5-1}dv = \int_1^5 \frac{v}{4}dv = [\frac{v^2}{8}]\Big|_1^5 = \frac{5^2}{8} - \frac{1^2}{8} = 3$$

   The expected willingness to pay of consumers is equal to 3. The monopolist extracts all the consumer surplus and sets its price equal to this: $p_2 = 3$.

   (b) What would be the optimal price that the monopolist charges in the second period if all consumers consumed in the first period (i.e. all consumers know their valuations)?

   Solution.

   Now every consumer knows their valuation. From the monopolists perspective the valuations are uniformly distributed between 1 and 5. Profits of the firm in the second period are given by:

$$\pi = p_2 Pr\{p_2 \leq v\} = p_2 * [1 - Pr\{p_2 > v\} = p_2[1 - F(p_2)] = p_2(1 - \frac{p_2 - 1}{5-1})$$
$$= p_2[\frac{4 - p_2 + 1}{4}] = \frac{5}{4}p_2 - \frac{1}{4}p_2^2$$

   Taking derivative of the profit function with respect to p_2 yields:

$$\frac{5}{4} - \frac{2}{4}p_2 = 0$$
$$p_2 = \frac{5}{2}$$

(c) Suppose that the monopolist charges in the second period the price that you derived in b). What is the maximum price that the consumers would then be willing to pay in the first period? (note: to compute this, you need to compute the value of information for the consumers)

Solution. Consumers know that $p_2 = \frac{5}{2}$. If the consumer does not buy in the first period then her expected payoff is: $\mathbb{E}[v] - \frac{5}{2} = \frac{1}{2}$. If she buys the good in the first period she gets:

$$\begin{aligned} \mathbb{E}[v] - p_1 + Pr\{v > \frac{5}{2}\}[\mathbb{E}[v|v > \frac{5}{2}] - \frac{5}{2}] \\ = 3 - p_1 + \frac{5}{8}\frac{5}{4} = \frac{121}{32} - p_1 \end{aligned}$$

Buying is preferred whenever:

$$\begin{aligned} \frac{121}{32} - p_1 \geq \frac{1}{2} \\ \frac{105}{32} \geq p_1 \end{aligned}$$

The maximum consumers are willing to pay is thus $p_1 = \frac{105}{32}$

(d) Using your findings above, characterize an equilibrium, where the monopolist chooses prices optimally in both periods, and the consumers choose optimally whether or not to buy, given the prices.

Solution.

The monopolist prefers that all consumers buy in both periods. In part (b) we showed that everyone buying in the first period yields optimal price $p_2 = \frac{5}{2}$ in the second period. This price p_2 then implies by part (c) that consumers are willing to pay $p_1 = \frac{105}{32}$ in the first period. So $p_1 = \frac{105}{32}, p_2 = \frac{5}{2}$ constitutes an equilibrium.

2. In this problem we continue with the example of Lecture 10, where we compared different procedures that a seller might use to sell an object. The seller has a single object to sell and there are just two potential buyers, whose valuations for the object are drawn independently from uniform distribution so that $v_i \sim U[0,1]$, $i = 1, 2$.

(a) Review the case, where the seller uses a poster price mechanism: the seller sets price p and buyers either accept or reject the offer. If one buyer accepts, she gets the object and pays p; if both buyers accept, the object is allocated randomly to one buyer who pays p; if neither accepts, the objects is not sold and no payments are made. What is the expected revenue to the seller for given p? Find the price that maximizes the expected revenue and compute that revenue.

Solution. The revenue of the seller is

$$\pi = p \times \Pr[\text{at least one } v \text{ above } p] = p \times [1 - \Pr(\text{both } v \text{ below } p)]$$
$$= p[1 - F(p)^2] = p[1 - p^2],$$

I have used the fact that for uniform $[0,1]$ distribution $F(p) = p$. Proceed by taking the first order condition of the revenue function with respect to price p. The FOC gives $p^* = \sqrt{1/3}$, thus expected revenue is $\sqrt{1/3}(1 - 1/3) = 2/(3\sqrt{3}) \approx 0.385$.

(b) Suppose the seller approaches the two buyers sequentially. First, she approaches buyer 1 and offers to sell at posted price p_1. If buyer 1 accepts, the object is sold at price p_1 and the game is over. If rejected, the seller approaches buyer 2 and offers to sell at posted price p_2. If buyer 2 accepts, the object is sold at price p_2; if rejected, the good is not sold. What is the expected revenue to the seller if she sets $p_1 = p_2 = p$ (for an arbitrary p)?

Solution. Let v_1 and v_2 denote valuations of buyers. The revenue is given by:

$$\begin{aligned}
\pi &= p_1 \Pr[v_1 \geq p_1] + p_2 \Pr[v_1 < p_1] \Pr[v_2 \geq p_2] \\
&= p_1[1 - F(p_1)] + p_2 F(p_1)[1 - F(p_2)] \\
&= p_1[1 - p_1] + p_2 p_1[1 - p_2] \\
&= p_1 - p_1^2 + p_1 p_2 - p_1 p_2^2
\end{aligned}$$

Imposing $p_1 = p_2 = p$ yields: $\pi = p - p^3 = p[1 - p^2]$. The objective function is the same as in a).

(c) What is the expected revenue in (b) if the seller sets p_1 and p_2 optimally? Compare to a) and discuss.

Solution. Again proceed by taking first order conditions of the profit function with respect to prices. FOCs give

$$1 - 2p_1 + p_2 - p_2^2 = 0 \qquad \text{and} \qquad p_1 - 2p_1 p_2 = 0$$

From the second we get $p_2^* = 1/2$, and plugging into the first gives $p_1^* = 5/8$. The expected revenue with these values is then $25/64 \approx 0.391$. Thus by setting two different prices optimally, the seller can increase expected revenue.

(d) Suppose that the seller uses a second-price auction to allocate the object and she set a reserve price $r > 0$. Is bidding one's own value a dominant strategy for the buyers?

Solution. As discussed in lecture slides, $b_i = c_i$ is dominant in SPA, and every player following that strategy constitutes an equilibrium. To see why, consider bidder i with valuation v_i and denote the highest bid submitted by other players by $B \equiv \max_{j \neq i}\{b_j\}$ (B is unknown to player i). Because the player values the object at v_i, she wants to win only if $v_i > B$; if $v_i < B$, she wants to lose (she is indifferent when $v_i = B$). Thus, her optimal strategy would guarantee her win if $v_i > B$ and guarantee her loss if $v_i < B$.

Bidding her own valuation, $b_i = v_i$, achieves exactly this. If $b_i > B$ she wins the auction, pays B, and gets payoff $v_i - B \geq 0$. Similarly if $b_i < B$, she loses but did not want to win in the first place because $v_i - B < 0$. Thus, $b_i = v_i$ is the payoff-maximizing bid for i *regardless of other players' submitted bids.* Bidding above B (below B) runs the risk of winning (losing) the auction when the agent does not want to win (lose). Therefore, bidding $b_i = v_i$ is a (weakly) dominant strategy for each player i in a SPA.

Having a reserve price in a 2nd price auction does not change the reasoning above.

(e) Compute the expected revenue to the seller for an arbitrary r (Hint: think about the three different cases in the slides for Lecture 10, and utilize them in computing the expectation)

Solution. The revenue of the seller is 0 if the valuation of both bidders is below the reserve price r. The revenue of the seller is r if the valuation of only one bidder exceeds the reserve price. Finally the revenue of the seller is $min(v_1, v_2)$ if the valuation of both buyers exceeds the reserve price.

In 3 e) we are given the PDF of the second highest valuation. Plugging $N = 2$ to this formula yields $f(v) = 2(1 - v)$. This needs to be scaled because we are considering the special case where both of the valuations exceed the reserve price thus we have: $f(v) = \frac{2(1-v)}{(1-r)^2}$. The expected value is then given by: $\mathbb{E}[min(v_1, v_2)] = \int_r^1 v\frac{2(1-v)}{(1-r)^2}dv = \frac{2r+1}{3}$.

Now we are ready to write down the expected revenue of the seller! It is given by:

$$\Pi = 0 * Pr(v_1 < r)Pr(v_2 < r) + r * Pr(v_2 > r)Pr(v_1 < r) + r * Pr(v_1 > r)Pr(v_2 < r)$$
$$+ \frac{2r+1}{3} * Pr(v_1 > r)Pr(v_2 > r)$$

Rearranging and using the properties of uniform distribution [0,1] we can write this in the following manner:

$$\Pi = 2r^2(1-r) + \frac{2r+1}{3}(1-r)^2$$

(f) Show that $r = 1/2$ is the optimal reserve price, and compute the expected revenue at that reserve price.

Solution. Taking the derivative of the profit function with respect to the reserve price r yields: $r^* = \frac{1}{2}$. Plugging this into the revenue function yields: $\Pi = \frac{5}{12}$.

3. Here we consider a second-price sealed bid auction (SPA) where N bidders have independent valuations drawn from some distribution F.

(a) Formulate this auction as a Bayesian game.

Solution.

A Bayesian game:

- A set of players: bidders $i \in \mathcal{I} = \{1, ..., N\}$
- A set of actions: bids, $b_i \in B_i = \mathbb{R}_+$ for all $i \in \mathcal{I}$
- Types for players: private valuations $v_i \in V_i = \mathbb{R}_+$
- Valuations drawn independently from F_i for each i
- Strategies that map types into actions: $s_i : V_i \to B_i$ for each $i \in \mathcal{I}$
- Payoffs $u_i(b_1, ..., b_N) = v_i - \max_{j \neq i} b_j$ if i has the highest bid; $u_i(b_1, ..., b_N) = 0$ otherwise (ignoring ties)

(b) Is bidding one's own valuation a dominant strategy in this game.

Solution.

See 2 d)

(c) Suppose that instead of a SPA the seller uses a third-price auction, where the highest bidder wins but pays the third-highest bid. Is bidding one's own valuation a dominant strategy? If not, would you expect bidders to bid higher or lower than their own valuation?

Solution. No. As a counterexample, consider the bidder with the second-highest bid and valuation, $b_{(2)} = v_{(2)}$. She does not win and receives zero surplus. By increasing her bid above $b_{(1)}$ she would win and receive $v_{(2)} - b_{(3)} > 0$. Thus, she wants to deviate whenever her valuation is the second-highest and therefore bidding her own valuation is not a dominant strategy.

(d) Consider again the SPA and assume that all the players are using their dominant strategies. Suppose from now on that valuations are from uniform distribution

$v_i \sim U[0,1]$. If player i has valuation v_i, what is her chance of winning the auction?

Solution. Because everybody bids their own valuation, we have

$$\Pr\{i \text{ wins}\} = \Pr\{i \text{ has highest valuation}\} = \Pr\{v_i > \max_{j \neq i} v_j\},$$

Probability of v_i being higher than a single uniform $[0,1]$ random variable v_j is just

$$\Pr\{v_i > v_j\} = F(v_i) = v_i,$$

due to the properties of uniform distribution. Because this needs to happen $N-1$ times and valuations are independent, the probability that i wins is $F(v_i)^{N-1} = v_i^{N-1}$.

(e) Compute the expected revenue for the seller (Hint: the density function of the second-highest valuation is $g(v) = N(N-1)(1-v)v^{N-2}$. How do you derive this? How do you use this to compute the expectation of the second highest valuation?)

Solution. Expected revenue equals the expected payments that bidders make to the seller. The expected revenue generated by SPA is

$$R_{SPA} = \mathbb{E}[\text{second-highest bid}] = \int_{-\infty}^{\infty} vg(v)dv = \int_0^1 vg(v)dv,$$

where $g(v)$ is the density is the second-highest bid (note that $f(v) = 0$ for $v < 0$ and $v > 1$). By using the definition given in the question, we get

$$\begin{aligned}
R_{SPA} &= \int_0^1 vN(N-1)F^{N-2}(v)f(v)(1-F(v))dv \\
&= N(N-1)\int_0^1 v^{N-1}(1-v)dv \\
&= N(N-1)\int_0^1 v^{N-1} - v^N dv \\
&= N(N-1)[\frac{1}{N} - \frac{1}{N+1}] \\
&= \frac{N-1}{N+1}
\end{aligned}$$

Intuitively, the expected revenue to the seller increases with the number of bidders.

4. Consider a second-price auction where the seller sets a participation fee for the bidders. Assume that each of two potential bidders has a privately known private valuation v_i that is uniformly distributed on $[0,1]$. The auctioneer may charge a fee f to each

bidder for participating in the auction. If only one bidder participates, she pays the participation fee f but gets the good for free. If nobody participates, no payments are made. If both participate, then both bidders pay f and the good is allocated according to a second price auction. Keep in mind that we have a second-price auction, and therefore, in the event where both bidders participate, they bid according to $b_i = v_i$, $i = 1, 2$.

(a) If $f = 0$ do you expect both bidders to participate? Compute the expected value from participating in the auction for a bidder with v_i.

Solution. This is a standard second-price auction, where it is optimal for all the players to bid their true valuations. Hence, the expected value from participating (given that there are only two bidders) is given by:

$$\mathbb{E}u_i(v_i) = \Pr(v_j < v_i)\{v_i - \mathbb{E}(v_j \mid v_j < v_i)\} = F(v_i)\left[v_i - \frac{1}{F(v_i)}\int_0^{v_i} xf(x)\,dx\right]$$

$$\begin{aligned}\mathbb{E}u_i(v_i) &= \Pr(\text{win auction})(v_i - \mathbb{E}(\text{price to pay})) \\ &= \Pr(v_j < v_i)\{v_i - \mathbb{E}(v_j \mid v_j < v_i)\} \\ &= F(v_i)(v_i - \frac{1}{2}v_i) \\ &= v_i(v_i - \frac{1}{2}v_i) = \frac{1}{2}v_i^2\end{aligned}$$

(b) Let $f > 0$. For each $v_i \in [0, 1]$, compute the value of participating in the auction for i if she believes that j participates if and only if $v_j \geq v_i$.

Solution. Note that when i believes that j participates the auction if and only if $v_j \geq v_i$, then i believes that she will get the good if and only if j does not participate, and hence i gets the good for free. Thus, i's expected value from participating is given by

$$\begin{aligned}\mathbb{E}u_i(v_i) &= \Pr(v_j < v_i)\,v_i - f \\ &= F(v_i)\,v_i - f \\ &= v_i^2 - f\end{aligned}$$

(c) Argue that for any $f > 0$, there is an equilibrium where v_i participates in the auction (and pays f) if and only if $v_i \geq v(f)$ and $v(f)$ solves:

$$v(f)^2 = f,$$

or

$$v(f) = \sqrt{f}.$$

Solution. We showed above that the payoff from participating is $v_i^2 - f$ if i

believes that j participates only when $v_j \geq v_i$. The payoff has "a threshold" at zero exactly when $v_i = \sqrt{f}$, and is positive for higher v_i. Thus there is a symmetric equilibrium where both participate if $v_i \geq \sqrt{f}$, as we wanted to show.

(d) (Harder) Compute the expected revenue to the seller and solve for the optimal participation fee f. Can you use the revenue equivalence theorem to explain the relationship of this result to the result in 2(f)?

Solution. We consider the equilibrium calculated above. The problem of the seller is

$$\max_f \Big\{ f \Pr(\text{only one participates})$$

$$+ [2f + \mathbb{E}(v_2 \mid \text{both participate})] \Pr(\text{both participate}) \Big\}.$$

Again, there is a quick an easy way to solve the problem if you know what you're doing. The above problem is

$$\max_f \{ f[2\sqrt{f}(1-\sqrt{f})] + (1-\sqrt{f})^2 [2f + \sqrt{f} + \frac{1}{3}(1-\sqrt{f})] \},$$

where expected value of the lower valuation if both participate is $\sqrt{f} + \frac{1}{3}(1 - \sqrt{f})$ by standard arguments used earlier in this course (properties of uniform distribution). Maximizing this gives $f^* = \frac{1}{4}$, and the resulting expected revenue is $\frac{5}{12}$. This equals the expected revenue of the seller in a second-price auction, when the reservation price is set optimally at $\frac{1}{2}$ (as has been shown in 3 d)). This can be explained by the revenue equivalence theorem. The optimal reserve price is $r = \frac{1}{2}$ and the optimal participation fee is $f = \frac{1}{4}$. With these optimal values for the reserve price and participation fee we get the same bidders participating in equilibrium and hence the probabilities of winning are equal. Thus, we have equivalent allocation rules in these auctions. This means the expected payoffs for bidders are the same, from which it follows that the expected revenue to the seller must also be equal.