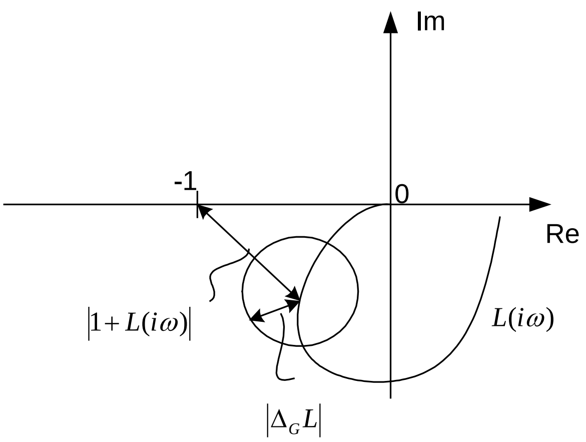
ELEC-E8116 Model-based control systems
/ exercises with solutions 6

Problem 1. Consider a SISO-system in a one-degree-of-freedom control configuration. The connection between the real and nominal system is

$$G_0(s) = G(s)(1+\Delta_G(s))$$

By using the Nyquist stability criterion derive a condition to the system to be robustly stable.

Solution. In the one-degree-of-freedom configuration $F_y = F_r$ and the loop transfer function of the nominal system is $L = GF_y$. The Nyquist curve is seen in the figure



At each point $L(i\omega)$ a circle with radius $L\Delta_G$ describes the model uncertainty such that the real curve is ceratinly inside the circle. Assuming that the nominal system is stable (no poles in RHP), the closed loop system is robustly stable exactly when the Nyquist curve does not encircle the critical point (-1,0). That can be expressed as (see figure)

$$|\Delta_G L| < |1+L|, \quad \forall\omega$$

and further

$$|T| < \frac{1}{|\Delta_G|}, \quad \forall\omega$$

which is the same as

$$\|\Delta_G T\|_\infty < 1$$

The result is the same as in the textbook formulas (6.28) and (6.29).

Problem 2. Consider the first order process

$$G_P(s) = \frac{k}{\tau s + 1} e^{-\theta s}$$

with parameter uncertainties such that $2 \le k, \theta, \tau \le 3$. The system is modelled with

$$G_0(s) = G(s)(1 + \Delta_G(s))$$

in which the nominal model is chosen to be the first order model without delay

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5s + 1}$$

Discuss possible candidates for the function $\Delta_G(s)$.

Solution.
To be exact, an accurate uncertainty area in each frequency of the complex function $G_P(i\omega)$ should be determined (corresponding to the parameter variations). This is very difficult, however, and in practice an approximate solution with uncertainty circles is used; see the solution to problem 1 and the equation

$$G_0(s) = G(s)(1 + \Delta_G(s))$$

The largest relative error must be determined

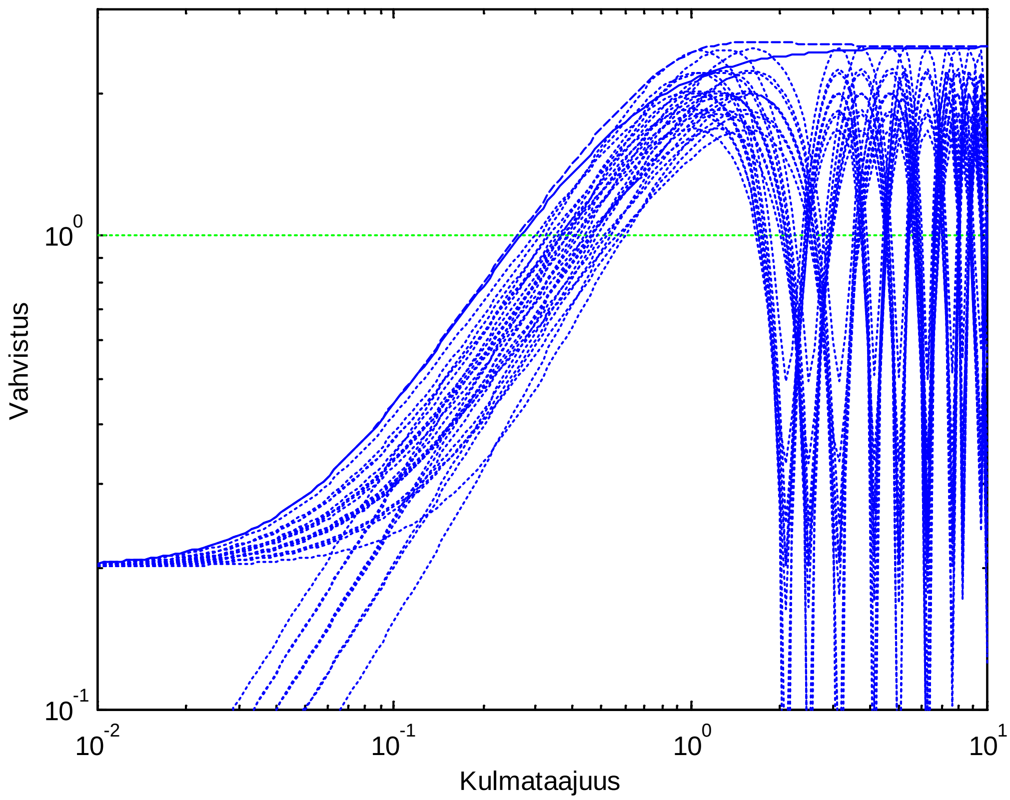
$$l_I(\omega) = \max_{G_P \in \Pi} \left| \frac{\mathrm{G_P}(i\omega) - G(i\omega)}{G(i\omega)} \right|$$

(Π means all possible models when the parameters vary within the given intervals).

Then we can take

$$|\Delta_G(i\omega)| \ge l_I(\omega), \quad \forall \omega$$

To calculate $l_I(\omega)$ choose the values 2, 2.5 and 3 for each variable (k, θ, τ). That does not necessarily describe the worst possible situation, but it is a step to the right direction. For the functions $l_I(\omega)$ we obtain $3^3 = 27$ curves, which are shown in the figure



The curve $l_I(\omega)$ must at each freqiency be larger than the dotted curves. It is seen that the value of $l_I(\omega)$ in small frequencies is 0.2 and 2.5 in large frequencies. For a candidate of Δ_G try a first order model, which corresponds to this behaviour

$$\Delta_{G1}(s) = \frac{Ts + 0.2}{(T/2.5)s + 1}, \quad T = 4$$

From the solid line it is seen that this is pretty good except near the frequency $\omega = 1$ where $\Delta_{G1}(s)$ is a little too small to cover all uncertainties. Increase the magnitude a bit near that particular frequency

$$\Delta_{G2}(s) = \Delta_{G1}(s)\frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1}$$

which is good (dash-dotted line in the figure).

Problem 3. Consider the process described in Exercise 5, Problem 1 with the exception that the process model is uncertain. The true system is

$$G_0(s) = G(s)(1 + \Delta_G(s))$$

in which the relative uncertainty has been modeled as

$$\Delta_G(s) = \frac{10s + 0.33}{(10/5.25)s + 1}$$

Is the controlled (closed loop) system robustly stable?

Solution.

The process model and controller were

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)} \qquad K(s) = 1.136(1 + \frac{1}{12.7s})$$

As noticed in problem 27 the determination of the relative error Δ_G is difficult. Often a simple error model is used, e.g. the first order transfer function

$$\Delta(s) = \frac{\tau s + r_0}{(\tau / r_\infty)s + 1}$$

where r_0 is the relative error of the stationary state, $1/\tau$ is approximately that angular frequency, in which the relative error reaches the 100% level, and r_∞ is the relative error in high frequencies (typically $r_\infty \geq 2$).

In the problem the relative error of the process model has been assumed to be 0.33 in small frequencies, about. 1 in frequency 0.1 *rad*/*s* and 5.25 in high frequencies.

The system is robustly stable, if it holds for all frequencies (textbook formula (6.29))

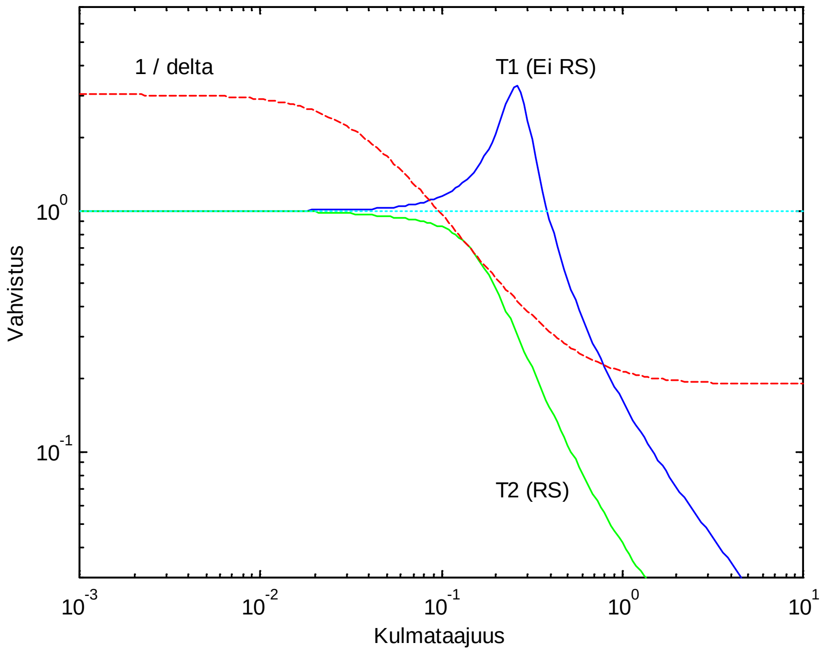
$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}$$

in which the complementary sensitivity function is

$$T = \frac{GK}{1 + GK}$$

From the figure it is seen that the system is not robustly stable, because the complementary sensitivity function $T1$ exceeds $1/\Delta_G$ in frequencies above 0.1.

By decreasing the gain of the PI-controller from 1.13 to 0.31 (trial and error result) a robustly stable closed-loop system is obtained (curve $T2$).



Problem 4. Let a closed-loop SISO-system be stable. Prove that the maximum delay that can be added to the process without causing closed-loop instability is

$$\theta_{\max} = PM / \omega_c$$

where PM is the phase margin of the (original) system and ω_c is the gain crossover frequency.

Solution.
Let G be the original transfer function. In the gain crossover frequency it holds

$$|G(i\omega_c)| = 1 \qquad \text{and} \qquad \arg G(i\omega_c) = -\pi + PM$$

When pure delay is added to the process

$$\left|G(i\omega_c)e^{-i\theta\omega_c}\right| = |G(i\omega_c)| = 1$$

which means that the gain crossover frequency remains the same. For the phase

$$\arg G(i\omega_c)e^{-i\theta\omega_c} = \arg G(i\omega_c) - \theta\omega_c = -\pi + PM - \theta\omega_c$$

At the stability limit

$$\arg G(i\omega_c)e^{-i\theta_{\max}\omega_c} = -\pi + PM - \theta_{\max}\omega_c = -\pi$$

from which

$$\theta_{\max} = \frac{PM}{\omega_c}$$