Complex Analysis, MS-C1300

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Lectures: Mondays & Thursday on campus

Office Hours (Fridays? 9-10?)
14-15?

Exercise Classes: All on campus

Hand-in 2 exercises before via MyCourses

Meeting 1: Tuesday or Wednesday
Deadline: Monday 23:59
(not 1^{st} week)

Meeting 2: Thursday or Friday
Deadline: Wednesday 23:59

You can collect all in all 6 bonus points

Exam: Max 24 points (Pass: 12 points)

Book: • An Introduction to Complex Function Theory, Palka

• Complex Analysis by Ahlfors is also very good but maybe more challenging.

The Complex Number System

We know that there are polynomials $p: R \to R$ that have no roots.

Basic example: $p(x) = x^2 + 1$ have no real root since $x^2 \geq 0$.

However if we add a number i with the property $i^2 = -1$ to the real numbers the polynomial $x^2 + 1$ have a root. When we add i to the real numbers then we get all numbers of the form

$$z = x + iy \qquad x, y \in \mathbb{R}$$

This might seem a little suspicious but can be done formally as follows.

Definition: The field of complex numbers

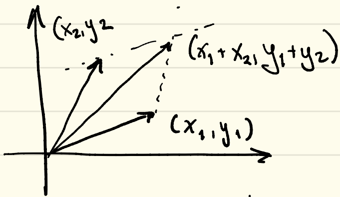
$\mathbb{C}$ consists of ordered pairs (x, y) $x, y \in \mathbb{R}$ with addition

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and multiplication

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

That is we can view complex numbers as points or vectors in $\mathbb{R}^2$ and addition is ordinary vector addition



We return to what multiplication means geometrically soon. First we state a theorem

Theorem 1 The complex number system is a field

since
- $\forall z, w \in \mathbb{C}$: $z + w = w + z$
- $\forall z, w \in \mathbb{C}$: $z \cdot w = w \cdot z$
- $\forall z_1, z_2, z_3 \in \mathbb{C}$: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $\forall z_1, z_2, z_3 \in \mathbb{C}$: $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$
- $\exists 0 \in \mathbb{C}$: $\forall z \in \mathbb{C}$: $z + 0 = z$
- $\exists 1 \in \mathbb{C}$: $\forall z \in \mathbb{C}$: $z \cdot 1 = z$
- $\forall z \in \mathbb{C}$: $\exists (-z) \in \mathbb{C}$: $z + (-z) = 0$
- $\forall z \in \mathbb{C} \setminus \{0\}$: $\exists z^{-1} \in \mathbb{C}$: $z \cdot z^{-1} = 1$
- $\forall z_1, z_2, w \in \mathbb{C}$: $w \cdot (z_1 + z_2) = w \cdot z_1 + w \cdot z_2$

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"In a field addition and multiplication is commutative. We can subtract by any number. We can divide by any non-zero number. Addition and multiplication is distributive"

These properties are easily verified (except maybe the existence of z^{-1}). We see that

$$0 = (0,0) \quad \text{and} \quad 1 = (1,0)$$

Given $z = (x,y) \neq (0,0)$ one checks that

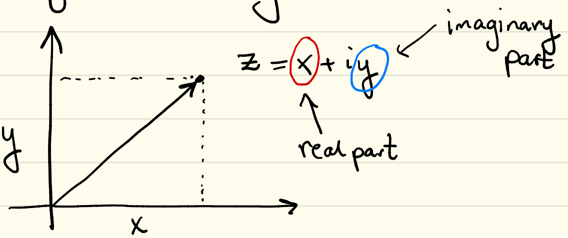
$$z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$$

However, calculating with complex numbers becomes easier when we realize that

$$(0,1) \cdot (0,1) = (-1,0) = -(1,0) = -1$$

so $i = (0,1)$. (Remember $i^2 = -1$)

Therefore we use $x + iy$ as short hand for $(x,y) = x(1,0) + y(0,1)$



Using this shorthand and remembering $i^2 = -1$ we easily calculates

<u>Ex</u>

$$(1+4i)+(-2+i) = -1+5i$$

$$(1+4i)(-2+i) = -2+1i-8i+4i^2 = -6-7i$$

Lets get some geometric intuition for multiplication

$z = x+iy$

$Re(z) = x$ = real part of z

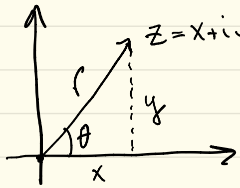
$Im(z) = y$ = imaginary part of z

$\overline{z} = x - iy$ = conjugate of z

Notice that $z\overline{z} = (x+iy)(x-iy) = x^2+y^2 \geq 0$

Also $Re(z) = \dfrac{z+\overline{z}}{2}$ and $Im(z) = \dfrac{z-\overline{z}}{2i}$

Define the modulus of z (or absolute value of z) as $|z| = \sqrt{z\overline{z}} = \sqrt{x^2+y^2}$



$$z = x+iy = r(\cos\theta + i\sin\theta) = \underbrace{= |z|(\cos\theta + i\sin\theta)}_{\text{polar form}}$$

We say that θ is <u>an</u> argument of z

Notice that if θ is an argument of z then

$\theta + 2\pi n$, $n = 0, \pm 1, \pm 2, \dots$ also is

Take two complex numbers z and w and write them in polar form.

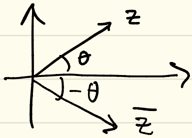
$$z = |z| (\cos\theta + i \sin\theta)$$
$$w = |w| (\cos\phi + i \sin\phi)$$

Multiply and we get

$$\begin{aligned} zw &= |z||w| (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \\ &= |z||w| \big((\cos\theta\cos\phi - \sin\theta\sin\phi) + \\ &\qquad + i(\cos\theta\sin\phi + \sin\theta\cos\phi) \big) = \\ &= |z||w| \big(\cos(\theta+\phi) + i\sin(\theta+\phi) \big) \end{aligned}$$

So you get the absolute value of the product it you multiply the absolute values of the factors and an argument it add arguments of the factors.

If $z = |z|(\cos\theta + i\sin\theta)$ it is clear that $\overline{z} = |z|(\cos(-\theta) + i\sin(-\theta))$ since



Also, since $zz^{-1}=1$ (if $z\neq 0$) we see that

$$z^{-1}=\frac{1}{z}=|z|^{-1}\left(\cos(-\theta)+i\sin(-\theta)\right).$$

We also have $\dfrac{z}{w}=\dfrac{z\overline{w}}{w\overline{w}}=\dfrac{1}{|w|^2}z\overline{w}$

Ex: $\dfrac{1-i}{1+i}=\dfrac{(1-i)(1-i)}{(1+i)(1-i)}=\dfrac{1-i-i+i^2}{1^2-i^2}=\dfrac{-2i}{2}=-i$

De Moivre's Formula

Assume $z=|z|(\cos\theta+i\sin\theta)$ Then

$$z^n=|z|^n(\cos\theta+i\sin\theta)^n=|z|^n\left(\cos(n\theta)+i\sin(n\theta)\right)$$

In particular we notice

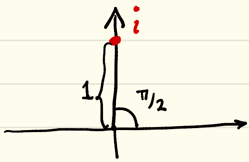
$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

(This holds for $n=0,\pm 1,\pm 2,\dots$)

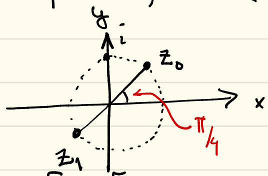
We can use the polar form to solve certain polynomial equations.

Ex Find all z such that $z^2=i$.

Write i in polar form.



We see that $i = \cos\left(\frac{\pi}{2}+2\pi n\right)+i\sin\left(\frac{\pi}{2}+2\pi n\right)$ and we look for $z = r(\cos\theta + i\sin\theta)$ such that $r^2 = 1$ and $2\theta = \frac{\pi}{2}+2\pi n$. Therefore $r = 1$ and $\theta = \frac{\pi}{4}+\pi n$; $n \in \mathbb{Z}$



$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ Both $z_0^2 = z_1^2 = i$

$z_1 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

If we want to define $\sqrt{i}$ (or $\sqrt{z}$ in general) we need to choose between z_0 and z_1. To make this choice we introduce the principal argument of z.
In general $\arg(z)$ is a set. Namely, if $z \neq 0$ then

$$z = |z|(\cos\theta + i\sin\theta) \text{ and } \arg(z) = \{\theta + 2\pi n ; n \in \mathbb{Z}\}$$

Arg(z)

The principal argument of z, $\mathrm{Arg}(z)$ is the element $\theta \in \arg(z)$ satisfying $-\pi < \theta \leq \pi$.

Principal n^{th}-root

Def: The principal n^{th} root of z

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\left(\frac{\mathrm{Arg}(z)}{n}\right) + i\sin\left(\frac{\mathrm{Arg}(z)}{n}\right)\right)$$

($\sqrt{z}$ is shorthand for $\sqrt[2]{z}$)

So for example i has 2 square roots ($\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$) but one principal square root $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Exponentials and logarithms

We want to define e^z where $z \in \mathbb{C}$. Recall from calculus Euler's formula $e^{iy}=\cos y + i\sin y$, $y\in\mathbb{R}$. This formula can be justified using theory coming later but for now we use it as a definition. We also define

$$\boxed{\exp(z)=e^z = e^{x+iy} = e^x\cdot e^{iy} = e^x(\cos y + i\sin y)}$$

Ex
$$e^{i\pi} = e^0(\cos\pi + i\sin\pi) = -1$$
$$e^{1+\frac{i\pi}{2}} = e^1\left(\cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}\right) = ei$$

Assume $z = x+iy$ and $w=u+iv$. We get

$$e^z\cdot e^w = e^x e^u(\cos y + i\sin y)(\cos v + i\sin v) =$$
$$= e^{x+u}\left((\cos y\cos v - \sin y\sin v) + i(\cos y\sin v + \sin y\cos v)\right)$$
$$= e^{x+u}\left(\cos(y+v)+i\sin(y+v)\right) = e^{z+w}$$

Also $e^{-z} = (e^z)^{-1} = \frac{1}{e^z}$

We also see that $|e^z| = e^{\operatorname{Re}(z)}$ and $\arg(e^z) = \operatorname{Im}(z) + 2\pi n$

Now lets study the equation $e^z = w$
(Solutions should be $\log w$)
Assume $w \neq 0$ and write $w = |w|\left(\cos(\operatorname{Arg} w) + i\sin(\operatorname{Arg} w)\right)$

Since $|w| > 0$ we write $|w| = e^{\ln|w|}$ using the ordinary $\ln : \mathbb{R}_{>0} \to \mathbb{R}$.

$$e^z = e^x(\cos y + i\sin y) = e^{\ln|w|}(\cos(\operatorname{Arg} w) + i\sin(\operatorname{Arg} w))$$

We see that any z of the form

$$z = \ln|w| + i(\operatorname{Arg} w + 2\pi n), \quad n \in \mathbb{Z}$$

solves $e^z = w$ when $w \neq 0$
(when $w = 0$ no solutions)

We are forced to accept that in general the logarithm is multi-valued!

If $z \neq 0$; $\log z = \ln|z| + i(\operatorname{Arg}(z) + 2\pi n)$; $n \in \mathbb{Z}$
$= \ln|z| + i\arg(z)$

Remember
$\arg(z)$ is a set

We also define the principal logarithm:

If $z \neq 0$ then $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$

Ex $\mathrm{Log}(-1) = \ln|-1| + i\,\mathrm{Arg}(-1) = 0 + i\pi = i\pi$

$$\log(-1) = \ln|-1| + i\arg(-1) = i(\pi + 2\pi n)$$
$n \in \mathbb{Z}$

Recall from calculus that $x^a = e^{a\ln x}$ is (one way) of defining x^a when $a \in \mathbb{R}$. Now that we have the logarithm defined for $z \in \mathbb{C}\setminus\{0\}$ we can define z^w for $z \in \mathbb{C}\setminus\{0\}, w \in \mathbb{C}$. Since the logarithm is multi-valued also z^w will be multi-valued in general (since $z^w := e^{w\log(z)}$)

Ex $i^i = e^{i\log i} = e^{i(\ln|1| + i(\frac{\pi}{2} + 2\pi n))} = e^{-(\frac{\pi}{2} + 2\pi n)}$
$n \in \mathbb{Z}$

Of course one could choose for example the principal logarithm and get a single-valued power.

This what we will do in this course and when we write (and say nothing else) z^w we mean the principal w-power of z. That is $z^w = e^{w \operatorname{Log}(z)}$.

Some comments: • If $n \in \mathbb{Z}$ then any n-power is equal to the principal n-power

$$z^n = e^{n(\ln|z| + i\operatorname{Arg}(z))} = e^{n(\ln|z| + i(\operatorname{Arg}(z) + 2\pi k))} \quad k \in \mathbb{Z}$$

• $\sqrt[n]{z} = z^{1/n}$

• Be very careful with generalizing laws like $(zw)^\alpha = z^\alpha w^\alpha$ to complex exponents!

Almost none hold in general

⌜All "problems" stem from the fact that $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$ is false in some cases⌟

Example:

$$\sqrt{z} \cdot \sqrt{w} \neq \sqrt{zw} \quad \text{in general}$$

Think about the following "paradox"

$$-1 = i^2 = \sqrt{(-1)} \cdot \sqrt{(-1)} = \sqrt{1} = 1$$

WRONG!

Observe: It is true that $z^{w_1} \cdot z^{w_2} = z^{w_1 + w_2}$ for complex exponents.