We show that $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly when $\mathrm{Re}(z) \geq \sigma > 1$. Study $|n^{-z}| = |(e^{\ln n})^{-z}| = n^{-\mathrm{Re}(z)} \leq n^{-\sigma}$

We know that $\sum_{n=1}^{\infty} n^{-\sigma}$ converges (it is a p-series with $p>1$) So Weierstrass M-test gives the result. Normal convergence follows and $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is analytic. This is the Riemann zeta-function (representation valid in $\mathrm{Re}(z)>1$)

We also get

$$\zeta'(z) = -\sum_{n=1}^{\infty} (\ln n)\, n^{-z} \quad \text{(with normal convergence)}$$

Taylor Series

A Taylor series is a function series of the following type. Take a sequence of complex numbers $(a_n)_{n=0}^{\infty}$ and $z_0 \in \mathbb{C}$. Then

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{is a Taylor series}$$

(or power series) centered at z_0 with coefficients (a_n). It is interesting to determine for which z this converges (and also to see what type of convergence we have). The following will be important.

Def The "number" $\rho \in [0,\infty) \cup \{\infty\}$ defined as

$$\rho = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} \quad \text{is called}$$

the radius of convergence of the Taylor series.

Convention: $\frac{1}{0} = \infty$ & $\frac{1}{\infty} = 0$

Ex $\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup(a_k;\ k\geq n)\right)$

If $\lim_{n\to\infty} a_n$ exists then $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$

$a_n = 1/n \qquad \limsup_{n\to\infty} \frac{1}{n} = 0$

$a_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $\quad \lim_{n\to\infty} a_n$ does not exist but

$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1.$

Another characterization

$\limsup_{n\to\infty} a_n = A$ if for every $\varepsilon>0$ exists N so that $a_n \leq A+\varepsilon$ if $n\geq N$ and there is $k\geq N$ so that $a_k \geq A-\varepsilon$.

38 **Theorem 38** Suppose that ρ is the radius of convergence of a Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ centered at z_0. The series diverges for any z satisfying $|z-z_0|>\rho$. If $\rho>0$, the series converges absolutely and normally in the disk $\Delta=\Delta(z_0,\rho)$, and $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic in Δ. The coefficients a_n satisfies

$$a_n = \frac{1}{n!} f^{(n)}(z_0).$$

Proof: $\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. If $z\in\mathbb{C}$ satisfies $|z-z_0| = r>\rho$

then, since $\frac{1}{r} < \frac{1}{\rho}$, we can find $n_j \to \infty$

so that $\sqrt[n_j]{|a_{n_j}|} > \frac{1}{r}$ or $|a_{n_j}| > \frac{1}{r^{n_j}}$. Therefore

$|a_{n_j}(z-z_0)^{n_j}| = |a_{n_j}|\,|z-z_0|^{n_j} > \frac{1}{r^{n_j}} r^{n_j} = 1$ and

the terms in the Taylor series does not approach 0. Hence the Taylor series diverges outside $\overline{\Delta}$.

Now if $\rho > 0$ and $|z - z_0| \le r < \rho$. Let $\varepsilon > 0$ be such that $\frac{1}{\rho} + \varepsilon < \frac{1}{r}$. Since $\frac{1}{\rho} = \limsup \sqrt[n]{|a_n|}$ we can find N such that

$$\sqrt[n]{|a_n|} < \frac{1}{\rho} + \varepsilon \text{ if } n > N.$$

Then $|a_n (z - z_0)^n| = |a_n| |z - z_0|^n = \left(\frac{1}{\rho} + \varepsilon\right)^n r^n = \alpha^n$ where $\alpha = r\left(\frac{1}{\rho} + \varepsilon\right) < 1$. Then using Weierstrass M-test ($\sum \alpha^n$ converges when $0 \le \alpha < 1$) we see that $\sum a_n (z - z_0)^n$ converges absolutely and normally in $\Delta(z_0, r)$.

$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic in $\Delta(z_0, r)$ since $a_k (z - z_0)^k$ are for $k = 0, 1, 2, \ldots$. Also $f(z_0) = a_0$. We can differentiate the series term-wise so $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ and $f'(z_0) = a_1$. Continue differentiating and by induction we get $a_n = \frac{1}{n!} f^{(n)}(z_0)$. ∅

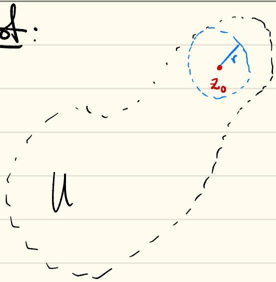
So we see that a Taylor series defines a function that is analytic inside $\Delta(z_0, \rho)$. We now show that any analytic function can be written as a Taylor series.

39 <u>Theorem 39</u> Assume that f is analytic in an open set U, that $z_0 \in U$ and that $\Delta = \Delta(z_0, r) \subseteq U$. Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in Δ and

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

(This means that the coefficients (a_n) are unique!)

Proof:



We want to show that $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$ in $\Delta=\Delta(z_0,r)$ where $a_n=\frac{1}{n!}f^{(n)}(z_0)$.

Pick $z\in\Delta$ and $0<s<r$ so that $z\in\Delta(z_0,s)$. By the Cauchy integral formula we have

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=s}\frac{f(\zeta)}{\zeta-z}\,d\zeta .$$

Also
$$\frac{f(\zeta)}{\zeta-z}=\frac{f(\zeta)}{\zeta-z_0}\cdot\frac{\zeta-z_0}{\zeta-z}=\frac{f(\zeta)}{\zeta-z_0}\cdot\frac{\zeta-z_0}{(\zeta-z_0)+(z_0-z)}=\frac{f(\zeta)}{\zeta-z_0}\,\frac{1}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}=$$
$$=\frac{f(\zeta)}{\zeta-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \quad \text{since } \left|\frac{z-z_0}{\zeta-z_0}\right|=\rho<1.$$

By Weierstrass M-test the series $\sum_{n=0}^{\infty}\frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}}$ (viewed as a function of ζ) converges uniformly on $|\zeta-z_0|=s$. Therefore, by Theorem 33,

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=s}\sum_{n=0}^{\infty}\frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}}\,d\zeta=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|\zeta-z_0|=s}\frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}}\,d\zeta$$
$$=\sum_{n=0}^{\infty}(z-z_0)^n\underbrace{\left(\frac{1}{2\pi i}\int_{|\zeta-z_0|=s}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\right)}_{=a_n}$$

Now by Cauchy's integral formula again

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_{|\zeta-z_0|=s}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta=n!\,a_n .$$

$\otimes$

Ex $f(z) = e^z$; $f^{(n)}(z) = e^z$

Therefore, since $f^{(n)}(0) = 1$, we get

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \text{when } |z| < \rho \text{. What is } \rho \text{?}$$

It has to be $\rho = \infty$! This is something that can be deduced from Theorem 39. There must be a point where f is not analytic on $|z - z_0| = \rho$! Since e^z is entire the series converges everywhere.

Ex $f(z) = \sin z$. Differentiation in the usual way gives

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{for } z \in \mathbb{C}$$

We can also differentiate Taylor series and get valid formulas (where convergence is uniform)

Hence

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

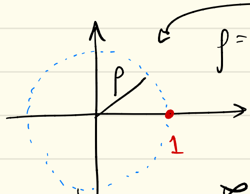
for $z \in \mathbb{C}$.

Ex $f(z) = e^{z^2}$. Differentiating here becomes tedious.

However since $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ and with $w = z^2$

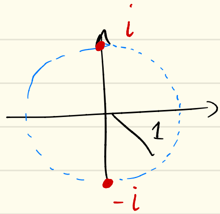
we get $e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$. This is a Taylor series and since they are unique we are done.

Ex $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. This is the Taylor series with center $z_0 = 0$. Where is the representation valid? When $|z|<1$ since $\rho = 1$.



Ex $f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

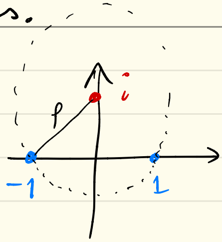
Valid when $|z|<1$ since



(This explains why the Taylor series for $f(x) = \frac{1}{1+x^2}$ behaves "strange". Also $\arctan(x)$)

Ex Expand $f(z) = \frac{2z}{z^2-1}$ in a Taylor series around $z_0 = i$. Also determine ρ for the series.

Solution:



$\rho = |i+1| = \sqrt{2}$

You can try to calculate $f^{(k)}(i)$ but this is tedious.
The reasonable method is to use geometric series.

$$\frac{2z}{z^2-1} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{(z-i)+(i-1)} + \frac{1}{(z-i)+(i+1)} =$$

$$= \frac{1}{(i-1)} \frac{1}{1-\left(-\frac{z-i}{i-1}\right)} + \frac{1}{(i+1)} \frac{1}{1-\left(-\frac{z-i}{i+1}\right)} =$$

$$= \frac{1}{(i-1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{i-1}\right)^n + \frac{1}{(i+1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{i+1}\right)^n =$$

$$= \sum_{n=0}^{\infty} \underbrace{(-1)^n \left[(i-1)^{-n-1} + (i+1)^{-n-1}\right]}_{= f^{(n)}(i)/n!} (z-i)^n$$

Consequences of the theory of Taylor series

40 **Theorem 40** (Identity principle)

If f is analytic in a domain D and if there exists a point $w_0 \in D$ such that $f^{(n)}(w_0) = 0$ for every integer $n \geq 1$ then f is constant in D.

Proof: Let $\mathcal{U} = \{ z \in D;\ f^{(n)}(z) = 0,\ n \geq 1 \text{ integer} \}$ and $V = D \setminus \mathcal{U}$. We show that $\mathcal{U}$ and V are both open and since $\mathcal{U} \neq \emptyset$ and $D = \mathcal{U} \cup V$ we will conclude that $D = \mathcal{U}$ and hence $f'(z) \equiv 0$ in D and f is constant. V is open since $f^{(k)}(z) \neq 0$ implies that $f^{(k)}(z) \neq 0$ in a open disk $\Delta(z,r) \subseteq V$ by continuity.

Now take $z_0 \in U$. We can then find $\Delta(z_0, r) \subseteq D$ and $f(z) = \sum_{n=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ in $\Delta(z_0, r)$.

Therefore $f(z)$ is constant in $\Delta(z_0, r)$ and we see that $\Delta(z_0, r) \subseteq U$ and hence U is open and non-empty. Hence $U = D$ and f is constant.

41 **Theorem 41** (The Factor Theorem)
Suppose that a function f is analytic and non-constant in a domain D and that z_0 is a point of D for which $f(z_0) = 0$. Then f can be uniquely represented in D in the fashion

$$f(z) = (z - z_0)^m g(z)$$

where m is a positive integer and $g : D \to \mathbb{C}$ is an analytic function such that $g(z_0) \neq 0$.

Proof: Since f is non-constant there is a minimal integer $m \geq 1$ so that $f^{(m)}(z_0) \neq 0$. Therefore the Taylor series around z_0 has the form

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n = (z - z_0)^m \sum_{k=0}^{\infty} a_{m+k}(z - z_0)^k$$

The function $g(z) = \begin{cases} \frac{f(z)}{(z - z_0)^m}, & \text{if } z \neq z_0 \\ a_m, & \text{if } z = z_0 \end{cases}$

is analytic in D. Also $g(z_0) = a_m \neq 0$.
Also g is unique since the Taylor series is unique.

The integer m in $f(z) = (z - z_0)^m g(z)$ is called the multiplicity (or order) of the zero at z_0.

Ex Determine the order of the zero of $f(z) = e^z - z - 1$ at $z_0 = 0$.

Solution: First we verify $f(0) = e^0 - 0 - 1 = 1 - 1 = 0$.

We calculate $f'(z) = e^z - 1$; $f'(0) = e^0 - 1 = 0$

$$f''(z) = e^z \quad ; \quad f''(0) = e^0 = 1 \neq 0$$

So the order of the zero is 2.

$\ulcorner f(z) = e^z - z - 1 = z^2 g(z)$; $g(0) \neq 0 \lrcorner$

Ex Determine the order of the zero of
$f(z) = \cos(z^3) - 1$.

Solution

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

$$f(z) = -1 + \cos(z^3) = -1 + 1 - \frac{z^6}{2!} + \frac{z^{12}}{4!} \ldots =$$

$$= -\frac{1}{2} z^6 + \frac{1}{4!} z^{12} - \ldots$$

Therefore $f(z)$ has a zero of multiplicity 6 at $z_0 = 0$.

$\ulcorner \cos(z^3) - 1 = z^6 g(z)$; $g(0) \neq 0 \lrcorner$

42

Theorem 42 (l'Hospital's Rule)
Let f and g be non-constant analytic functions in a disk $\Delta(z_0, r)$. Assume that $f(z_0) = g(z_0) = 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

(meaning that if one does not exist neither does the other.)

Proof: $f(z) = (z - z_0)^m h_1(z) \quad ; \quad h_1(z_0) \neq 0$
$g(z) = (z - z_0)^k h_2(z) \quad ; \quad h_2(z_0) \neq 0$

Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} (z - z_0)^{m-k} \frac{h_1(z)}{h_2(z)} = \begin{cases} 0 & \text{if } m > k \\ \frac{h_1(z_0)}{h_2(z_0)} & \text{if } m = k \\ \text{undefined} & \text{if } m < k \end{cases}$$

Also $f'(z) = m(z - z_0)^{m-1} h_1(z) + (z - z_0)^m h_1'(z)$ and
$g'(z) = k(z - z_0)^{k-1} h_2(z) + (z - z_0)^k h_2'(z)$.

We get

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \lim_{z \to z_0} (z - z_0)^{m-k} \frac{m h_1(z) + (z - z_0) h_1'(z)}{k h_2(z) + (z - z_0) h_2'(z)} = \begin{cases} 0 & \text{if } m > k \\ \frac{h_1(z_0)}{h_2(z_0)} & \text{if } m = k \\ \text{undefined} & \text{if } m < k \end{cases}$$

$$\Longrightarrow \lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)} \qquad \otimes$$

Ex Calculate $\lim_{z\to 0} \dfrac{z\cos z - z - 2z^2}{e^{z^2}-1}$

Solution: $f(z) = z\cos z - z - 2z^2 \qquad f(0)=0$

$g(z) = e^{z^2} - 1 \qquad g(0) = 0$

l' Hospital $\Rightarrow \lim_{z\to 0} \dfrac{f(z)}{g(z)} = \lim_{z\to 0} \dfrac{f'(z)}{g'(z)}$

$f'(z) = \cos z - z\sin z - 1 - 4z \qquad f'(0) = 0$

$g'(z) = 2ze^{z^2} \qquad g'(0) = 0$

$$\lim_{z\to 0}\frac{f(z)}{g(z)} = \lim_{z\to 0}\frac{f'(z)}{g'(z)} = \lim_{z\to 0}\frac{f''(z)}{g''(z)}$$

$f''(z) = -\sin z - \sin z - z\cos z - 4 \qquad f''(0) = -4$

$g''(z) = 2e^{z^2} + 4z^2e^{z^2} \qquad g''(0) = 2$

So $\lim_{z\to 0} \dfrac{z\cos z - z - 2z^2}{e^{z^2}-1} = -\dfrac{4}{2} = -2$

Laurent series

Doubly infinite series: $\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n$

$= (z_{-1} + z_{-2} + \dots\ ) + (z_0 + z_1 + \dots\ )$