Ex Calculate $\lim_{z\to 0} \dfrac{z\cos z - z - 2z^2}{e^{z^2}-1}$

Solution: $f(z) = z\cos z - z - 2z^2 \qquad f(0)=0$

$g(z) = e^{z^2} - 1 \qquad g(0)=0$

l'Hospital $\Rightarrow \lim_{z\to 0}\dfrac{f(z)}{g(z)} = \lim_{z\to 0}\dfrac{f'(z)}{g'(z)}$

$f'(z) = \cos z - z\sin z - 1 - 4z \qquad f'(0)=0$

$g'(z) = 2ze^{z^2} \qquad g'(0)=0$

$$\lim_{z\to 0}\frac{f(z)}{g(z)} = \lim_{z\to 0}\frac{f'(z)}{g'(z)} = \lim_{z\to 0}\frac{f''(z)}{g''(z)}$$

$f''(z) = -\sin z - \sin z - z\cos z - 4 \qquad f''(0) = -4$

$g''(z) = 2e^{z^2} + 4z^2 e^{z^2} \qquad g''(0) = 2$

So $\lim_{z\to 0}\dfrac{z\cos z - z - 2z^2}{e^{z^2}-1} = -\dfrac{4}{2} = -2$

Laurent series

Doubly infinite series: $\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n$

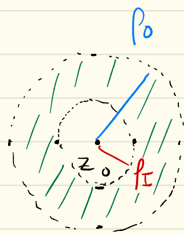
$= (z_{-1} + z_{-2} + \dots \quad) + (z_0 + z_1 + \dots \quad)$

Definition: A Laurent series centered at $z_0 \in \mathbb{C}$ is a doubly infinite function series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

In general a Laurent serie converges normally in sets $\{z \in \mathbb{C};\ 0 \leq \rho_I < |z-z_0| < \rho_0 \leq \infty\}$ where

$$\rho_0 = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} \quad \text{and} \quad \rho_I = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$$



Look at $\sum_{n=0}^{\infty} a_n (z-z_0)^n$. This defines an analytic function in $|z-z_0| < \rho_0$. We write $g_0(z-z_0) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. Now $g_I(w) = \sum_{n=1}^{\infty} a_{-n} w^n$

defines an analytic function in $|w| < \rho_I^{-1}$

Therefore

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = g_0(z-z_0) + g_I\left(\frac{1}{z-z_0}\right)$$

in $0 \leq \rho_I < |z-z_0| < \rho_0 \leq \infty$ (if $\rho_I < \rho_0$)

The convergence is normal in $\{\rho_I < |z-z_0| < \rho_0\}$ so f is analytic

Moreover, we can check that

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

if $\rho_I < r < \rho_0$.

This is true since

$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{1}{(z-z_0)^{n+1}} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k =$$

$$= \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k-n-1} \quad \text{and}$$

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} a_k \int (z-z_0)^{k-n-1} dz =$$

$$= \frac{1}{2\pi i} a_n \cdot 2\pi i = a_n .$$

We can also prove that any analytic function f in $0 \le a < |z-z_0| < b \le \infty$ can be written as a Laurent series. Namely

43 **Theorem 43**

Suppose that a function f is analytic in an annulus $D = \{z \in \mathbb{C};\ 0 \le a < |z-z_0| < b \le \infty\}$. Then f can be represented as a Laurent series centered at z_0. The coefficients a_n in

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{are unique.}$$

We have

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

for any r satisfying $a < r < b$.

The formulas $a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ are usually of little use practically.

Ex Expand $f(z) = e^{1/z}$ in a Laurent series in $D = \{z;\ 0 < |z| < \infty\}$.

Solution: We know that $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ for $w \in \mathbb{C}$.

Put $w = 1/z$. Then

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!} = 1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \cdots$$

This must be the Laurent series by uniqueness.

Ex Determine the Laurent series representation of $f(z) = \frac{\sin \pi z}{(z-1)^3}$ in $D = \{z \in \mathbb{C};\ 0 < |z-1| < \infty\}$

Solution:

First we develop $\sin \pi z$ in a Taylor series around $z = 1$

$g(z) = \sin \pi z \qquad g(1) = \sin \pi = 0$

$g'(z) = \pi \cos \pi z \qquad g'(1) = \pi \cos \pi = -\pi$

$g''(z) = -\pi^2 \sin \pi z \qquad g''(1) = 0$

$\vdots$

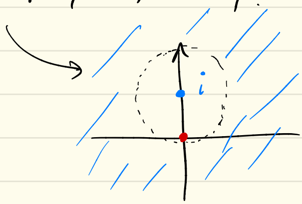
$g^{(2k)}(z) = (-1)^k \pi^{2k} \sin \pi z \qquad g^{(2k)}(1) = 0$

$g^{(2k+1)}(z) = (-1)^k \pi^{2k+1} \cos \pi z$

$g^{(2k+1)}(1) = (-1)^{k+1} \pi^{2k+1}$

So
$$\sin \pi z = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \pi^{2k+1}}{(2k+1)!} (z-1)^{2k+1}$$
when $z \in \mathbb{C}$. So
$$\frac{\sin \pi z}{(z-1)^3} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \pi^{2k+1}}{(2k+1)!} (z-1)^{2k-2}$$
$$= -\frac{\pi}{(z-1)^2} + \frac{\pi^3}{3!} - \frac{\pi^5}{5!}(z-1)^2 + \dots$$

<u>Ex</u> Develop $f(z) = \frac{1}{z}$ and $g(z) = \frac{1}{z^2}$ in Laurent series in the annulus $D = \{z ;\ 1 < |z-i| < \infty\}$.



Solution

$$\frac{1}{z} = \frac{1}{i + (z-i)} = \frac{1}{i}\left(\frac{1}{1 - i(z-i)}\right)$$

looks promising but doesn't work since $|i(z-i)| > 1$ in D

$$\boxed{\text{It works when } |z-i| < 1}$$

But $\frac{1}{z} = \frac{1}{(z-i)+i} = \frac{1}{(z-i)}\frac{1}{1-\frac{-i}{z-i}}$ works since $\left|\frac{i}{z-i}\right|<1$!

So $\frac{1}{z} = \frac{1}{z-i}\sum_{n=0}^{\infty}\frac{(-i)^n}{(z-i)^n} = \sum_{n=0}^{\infty}\frac{(-i)^n}{(z-i)^{n+1}} =$

$= \sum_{n=-\infty}^{-1}\frac{1}{i^{-n-1}}(z-i)^n = \sum_{n=-\infty}^{-1} i^{n+1}(z-i)^n$

$-i = \frac{1}{i}$

Finally $\frac{d}{dz}\frac{1}{z} = -1\frac{1}{z^2} = -\frac{1}{z^2}$ so

$-\frac{1}{z^2} = \sum_{n=-\infty}^{-1} n\, i^{n+1}(z-i)^{n-1} = \sum_{n=-\infty}^{-2}(n+1)\, i^{n+2}(z-i)^n$

or $\frac{1}{z^2} = \sum_{n=-\infty}^{-2}(n+1)\, i^n (z-i)^n$

(Isolated) Singularities of analytic functions

Let $\Delta^*(z_0,r) = \{z\in\mathbb{C};\ 0<|z-z_0|<r\}$

We call $\Delta^*(z_0,r)$ a punctured disk

Let $f:\Delta^*(z_0,r)\to\mathbb{C}$ be analytic.

We want to study how f can fail to be analytic at z_0.

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Removable singularities

If we can defined $f(z_0)$ so f becomes analytic in z_0 we say that the singularity is removable.

Theorem 44 (Riemann's Extension Theorem) Let f be analytic in $\Delta^*(z_0, r)$. Then z_0 is a removable singularity iff f is bounded in $\Delta^*(z_0, s) \cup |z-z_0| = s$ for some $s < r$.

"Proof" Expand f in a Laurent series in $\Delta^*(z_0, r)$.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \text{ where}$$

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z-z_0)^{n+1}} dz. \text{ Hence}$$

$$|a_n| \leq \frac{1}{2\pi} \frac{M}{s^{n+1}} 2\pi s = \frac{M}{s^n}. \text{ If } n < 0$$

then $|a_n| \leq M s^{-n} \to 0$ as $s \to 0$

Hence $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic in $\Delta(z_0, r)$.

$\limsup\limits_{z \to z_0} |f(z)| = ?$

$\lim\limits_{r \to 0^+} \left(\sup \left(|f(z)|;\ |z - z_0| < r \right) \right)$

$= \limsup\limits_{z \to z_0} |f(z)|$

So z_0 is removable if $\limsup\limits_{z \to z_0} |f(z)| < \infty$

Therefore z_0 is a "true" singularity if $\limsup\limits_{z \to z_0} |f(z)| = \infty$. Either $\lim\limits_{z \to z_0} |f(z)| = \infty$ or $\lim\limits_{z \to z_0} |f(z)|$ doesn't exist. If the limit $\lim\limits_{z \to z_0} |f(z)|$ doesn't exist we say that z_0 is an essential singularity. If $\lim\limits_{z \to z_0} |f(z)| = \infty$ then z_0 is called a pole.

Poles

Assume that $f: \Delta^*(z_0, r) \to \mathbb{C}$ has a pole at z_0. Since $\lim\limits_{z \to z_0} |f(z)| = \infty$ we can assume that $f(z) \neq 0$ in $\Delta^*(z_0, r)$.

Then $\frac{1}{f}$ is analytic in $\Delta^*(z_0, r)$ and also $\lim\limits_{z \to z_0} \left|\frac{1}{f}\right| = \lim\limits_{z \to z_0} \frac{1}{|f|} = 0$ so $\frac{1}{f}$ is analytic in $\Delta(z_0, r)$ (by Riemann's Extension Theorem). Also $\frac{1}{f(z_0)} = 0$ and the Factor Theorem says $\frac{1}{f(z)} = (z - z_0)^m h(z)$ where h is analytic and $h(z_0) \neq 0$. Rearranging we get

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{where } g(z) = \frac{1}{h(z)}$$

Expanding $g(z)$ in a Taylor series we get

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=-m}^{\infty} a_{n+m} (z-z_0)^n$$

$$= \underbrace{\frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots + \frac{a_{m-1}}{z-z_0}}_{\text{singular part at } z_0} + a_m + a_{m+1}(z-z_0) + \cdots$$

We say that f has a pole of order m at z_0.

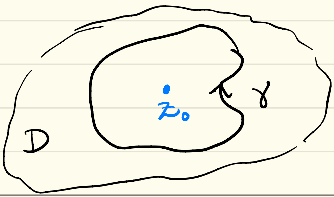
If f has an essential singularity at z_0 then the singular part is of "infinite order"

$$\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

The Residue Theorem and Residue Calculus

Say that we want to calculate $\int_\gamma f(z)\,dz$

where γ is as in the figure and f is analytic in $D \setminus \{z_0\}$.



If we knew the Laurent expansion of f around z_0 then we would have $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ and

$$\int_\gamma f(z)\,dz = \int_\gamma \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n\,dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz$$

$$= 2\pi i\, a_{-1} \quad \text{since} \quad \int_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n=-1 \end{cases}$$

The coefficient a_{-1} is called the residue of f at z_0. We write $a_{-1} = \operatorname{Res}(z_0, f)$.

<u>Ex</u> $\displaystyle\int_{|z|=1} e^{1/z}\,dz = 2\pi i \operatorname{Res}(0, e^{1/z})$ (Assume $|z|=1$ positively oriented)

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots + \frac{1}{n!\,z^n} + \cdots$$

So $\operatorname{Res}(0, e^{1/z}) = 1$

and $\displaystyle\int_{|z|=1} e^{1/z}\,dz = 2\pi i$

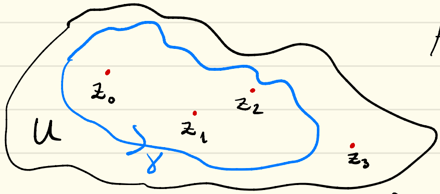
What about $\displaystyle\int_{|z|=1} z^k e^{1/z}\,dz = ?$

$$\int_{|z|=1} z^k e^{1/z}\,dz = 2\pi i \operatorname{Res}(0, z^k e^{1/z}) = 2\pi i \frac{1}{(k+1)!}$$

This seems quite powerful however two problems should be addressed:

- How do we handle several singularities?
- How to avoid Laurent series expansion?

Lets handle several singularities first.



Assume f has singularities at $z_0, z_1, z_2,$ and z_3.

Let S_n be the singular part of the Laurent series expansion at z_n. Observe that $f - S_0 - S_1 - S_2 - S_3$ is analytic in U!
Then Cauchy's Theorem tells us that

$$\int_\gamma f - S_0 - S_1 - S_2 - S_3 \, dz = 0$$

So

$$\int_\gamma f \, dz = \sum_{n=0}^{3} \int_\gamma S_n \, dz = \quad \text{in this case}$$

$$= 2\pi i \left(\mathrm{Res}(z_0, f) + \mathrm{Res}(z_1, f) + \mathrm{Res}(z_2, f) \right)$$

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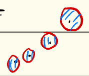
Theorem 45 (The Residue Theorem)

Assume that f is analytic in $U \setminus E$ where E is a discrete countable set (z_k) (finite or countably infinite) in U.

[discrete = points are isolated]

=



Assume σ is a cycle in $U \setminus E$ that is homologous to zero in U. Then

$$\int_\sigma f(z) \, dz = 2\pi i \sum_k n(\sigma, z_k) \, \mathrm{Res}(z_k, f)$$

Now, let's turn to the second question. How do we avoid Laurent expansions? It is not always possible. However, for poles we can find a formula.

Say that z_0 is a pole of order m for $f(z)$ then

Residue

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + \dots$$

$$(z-z_0)^m f(z) = a_{-m} + \dots + a_{-1}(z-z_0)^{m-1} + \dots$$

$$\frac{d}{dz}\left[(z-z_0)^m f(z)\right] = a_{-m+1} + \dots + (m-1)a_{-1}(z-z_0)^{m-2}$$

$$\left(\frac{d}{dz}\right)^{m-1}\left[(z-z_0)^m f(z)\right] = (m-1)!\,a_{-1} + \dots$$

So we have a formula

$$\operatorname{Res}(z_0, f) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Notice that you need to know the order of the pole to use this formula.

This you find out by finding the order of the zero z_0 for $1/f$.