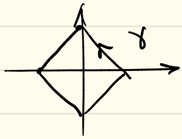
Evaluating Integrals with the Residue Theorem

Ex Evaluate $\int_\gamma \frac{z}{e^z - 1}\,dz$ where $|\gamma| = \{|x|+|y|=1\}$ (positively oriented)

Solution:



$f(z) = \frac{z}{e^z-1}$ is defined an analytic when $e^z \neq 1$.

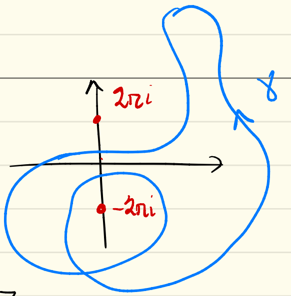
That is, $z_k = 2i\pi k$, $k \in \mathbb{Z}$. Only $z_0 = 0$ has $n(\gamma, 0) \neq 0$. Is $z_0 = 0$ a singularity for f?

We check $\lim_{z\to 0} f(z) = \lim_{z \to 0} \frac{z}{e^z - 1} \underset{\text{l'Hospital}}{=} \lim_{z\to 0} \frac{1}{e^z} = 1$

Hence $\tilde{f}(z) = \begin{cases} f(z) & z \neq 0 \\ 1 & z = 0 \end{cases}$ is analytic inside γ and

$$\int_\gamma \frac{z}{e^z-1}\,dz = 0 \text{ by Cauchy's Theorem}$$

Ex Change γ to the one in the figure. Calculate $\int_\gamma \frac{z}{e^z-1}\,dz$.



$n(\gamma, -2\pi i) = 2$

$$\Longrightarrow \int_\gamma \frac{z}{e^z - 1}\,dz = 2\pi i \cdot 2 \cdot \operatorname{Res}\left(-2\pi i, \frac{z}{e^z-1}\right)$$

Since $g = \frac{1}{f} = \frac{e^z - 1}{z}$ and $g(-2\pi i) = 0$, $g'(z) = \frac{ze^z - (e^z - 1)}{z^2}$,

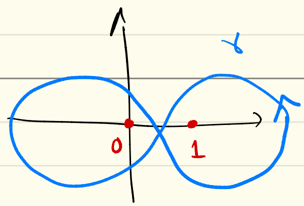
and $g'(-2\pi i) = \frac{-2\pi i}{-4\pi^2} \neq 0$ we see that $(-2\pi i)$ is a simple pole for f. So

$$\operatorname{Res}\left(-2\pi i, \frac{z}{e^z - 1}\right) = \lim_{z \to -2\pi i} \frac{z(z + 2\pi i)}{e^z - 1} =$$

$$\underset{\text{l'Hospital}}{=} \lim_{z \to -2\pi i} \frac{2z + 2\pi i}{e^z} = -2\pi i$$

The residue theorem gives $\int_\gamma \frac{z}{e^z - 1}\,dz = -4\pi i$

Ex Calculate $\int_\gamma \frac{\cos \pi z}{z(z-1)}\,dz$ where



$f(z) = \dfrac{\cos \pi z}{z(z-1)}$ has singularities at $z=0$ and $z=1$

We see that $n(\gamma,0)=-1$ and $n(\gamma,1)=1$.

We see that both singularities are simple poles,

$\operatorname{Res}(0,f(z)) = \lim_{z\to 0} z f(z) = \dfrac{\cos 0}{-1} = -1$, and

$\operatorname{Res}(1,f(z)) = \lim_{z\to 1} (z-1) f(z) = \dfrac{\cos \pi}{1} = -1$.

$$\Rightarrow \int_\gamma \frac{\cos \pi z}{z(z-1)}\,dz = 2\pi i\left((-1)\cdot(-1) + 1\cdot(-1)\right) = 0$$

$\otimes$

Remember that $x = \dfrac{z+\bar z}{2}$ and $y = \dfrac{z-\bar z}{2i}$. Also note that when $|z|=1$ then $\bar z = \dfrac{1}{z}$ (since $|z|^2 = z\bar z = 1 = z\cdot\frac{1}{z}$)

This can be exploited when we integrate

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$$ where $R(x,y)$ is a rational function.

This is because when $z = e^{i\theta} = \cos\theta + i\sin\theta$ then

$\cos\theta = \dfrac{z+\bar z}{2} = \dfrac{z+\frac{1}{z}}{2}$ and $\sin\theta = \dfrac{z-\bar z}{2i} = \dfrac{z-\frac{1}{z}}{2i}$. Also note that $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$.

[$t = \tan(\theta/2)$ also gives a way to solve these but often with more work]

Ex Calculate $\int_0^{2\pi} \sin^2\theta\,d\theta$

Solution: (This can be solved using calculus methods also.)

$$\int_0^{2\pi} \sin^2\theta\,d\theta = \int_{|z|=1} \left(\frac{z-\frac{1}{z}}{2i}\right)^2 \frac{1}{iz}\,dz = -\frac{1}{4i}\int_{|z|=1} \frac{1}{z}\left(z-\frac{1}{z}\right)^2 dz$$

$$= -\frac{1}{4i}\int_{|z|=1} \frac{(z^2-1)^2}{z^3}\,dz = -\frac{1}{4i}\int_{|z|=1} \frac{z^4-2z^2+1}{z^3}\,dz =$$

$$= -\frac{1}{4i}\,2\pi i\,(-2) = \pi$$

Ex Calculate $\int_0^{2\pi} \frac{1}{a+\cos\theta}\, d\theta$, where $a>1$.

Solution: We use the same trick.

$$\int_0^{2\pi} \frac{1}{a+\cos\theta}\, d\theta = \int_{|z|=1} \frac{1}{iz} \frac{1}{a+\frac{(z+z^{-1})}{2}}\, dz = \int_{|z|=1} \frac{1}{iz} \frac{2}{2a+(z+\frac{1}{z})}\, dz =$$

$$= \frac{1}{i} \int_{|z|=1} \frac{2}{z^2+2az+1}\, dz$$

$$z^2+2az+1 = (z+a)^2+1-a^2 = 0 \iff z+a = \pm\sqrt{a^2-1}$$

$$\iff z_1 = -a+\sqrt{a^2-1} \quad \text{and} \quad z_2 = -a-\sqrt{a^2-1}$$

z_1 is inside $|z|=1$

$$\operatorname{Res}\left(-a+\sqrt{a^2-1}, \frac{2}{z^2+2az+1}\right) = \lim_{z\to z_1} \frac{(z-z_1)2}{(z-z_1)(z-z_2)} = \frac{2}{z_1-z_2} = \frac{1}{\sqrt{a^2-1}}$$

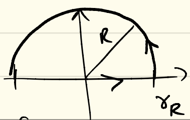
$$\implies \int_0^{2\pi} \frac{1}{a+\cos\theta}\, d\theta = \frac{1}{i}\, 2\pi i \cdot \frac{1}{\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

Ex We know how to calculate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

using $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. We can also

use residue calculus. We calculate $\int_{\gamma_R} \frac{1}{1+z^2}\, dz$

where



$\gamma_R = [-R,R] + C(R)$

If $R>1$ then $\int_{\gamma_R} \frac{1}{1+z^2}\, dz = 2\pi i \operatorname{Res}\left(i, \frac{1}{1+z^2}\right) = \pi$

since $\operatorname{Res}\left(i, \frac{1}{1+z^2}\right) = \lim_{z\to i} (z-i)\frac{1}{1+z^2} = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$

On the semicircle $c(R)$ we have

$$\left| \int_{c(R)} \frac{1}{1+z^2}\, dz \right| \leq \frac{\pi R}{R^2-1}$$

Let $R \to \infty$ and we see

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$$

However, the residue theorem lets us do more.

Ex

Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx.$

Use the same contour as before. On $c(R)$ we have

$$\left| \int_{c(R)} \frac{1}{(z^2+1)(z^2+4)}\, dz \right| \leq \frac{\pi R}{(R^2-1)(R^2-4)}$$

and for $R>2$ the residue theorem gives

$$f(z) = \frac{1}{(z^2+1)(z^2+4)}$$

$$\int_{\gamma_R} \frac{1}{(z^2+1)(z^2+4)}\, dz = 2\pi i \left(\mathrm{Res}(i, f(z)) + \mathrm{Res}(2i, f(z)) \right)$$

Since $\displaystyle \mathrm{Res}(i, f(z)) = \lim_{z\to i} (z-i) f(z) = \lim_{z \to i} \frac{1}{(z+i)(z^2+4)} = \frac{1}{2i\cdot 3}$,

and $\displaystyle \mathrm{Res}(2i, f(z)) = \lim_{z\to 2i} (z-2i) f(z) = \lim_{z\to 2i} \frac{1}{(z^2+1)(z+2i)} = \frac{1}{-3\cdot 4i}$

we get

$$\int_{\gamma_R} \frac{1}{(z^2+1)(z^2+4)}\, dz = 2\pi i \left(\frac{1}{6i} - \frac{1}{12i} \right) = \frac{\pi}{6}.$$

Therefore, after letting $R \to 0$, we get

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6}$$

Given $f : \mathbb{R} \to \mathbb{R}$ we can define it's Fourier transform $\mathcal{F}f : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{F}f(p) = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

(There are many slightly different definitions of $\mathcal{F}f$. Also many times $\mathcal{F}f$ is defined for p being complex.)

<u>Ex</u> Express the Fourier transform of $f(x) = \frac{1}{x^2+1}$ in closed form.

<u>Solution</u>: $\mathcal{F}f(p) = \int_{-\infty}^{\infty} \frac{e^{ipx}}{x^2+1}\, dx$. We use the

residue theorem but we need to be a little careful depending on if $p \geq 0$ or $p < 0$.
Assume $p \geq 0$. We study $g(z) = \frac{e^{ipz}}{z^2+1}$ on

γ_R from before. We are interested in $|e^{ipz}|$ on $C(R)$ (also from before).

$$|e^{ipz}| = e^{Re(ipz)} = e^{-p\,Im(z)} \leq e^{-p} \quad \underbrace{\text{on } C(R)}_{Im(z) \geq 0}$$

So $\left| \int_{C(R)} \frac{e^{ipz}}{z^2+1}\, dz \right| \leq \frac{\pi e^{-p} R}{R^2 - 1}$.

Also

$$\int_{\gamma_R} \frac{e^{ipz}}{z^2+1}\, dz = 2\pi i\, Res\left(i, \frac{e^{ipz}}{(z^2+1)}\right) = 2\pi i \lim_{z \to i} \frac{e^{ipz}}{z+i} =$$

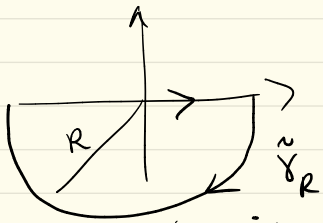
$$= 2\pi i \frac{e^{-p}}{2i} = \pi e^{-p}$$

Let $R \to \infty$ and we get

$$\mathcal{F}f(p) = \int_{-\infty}^{\infty} \frac{e^{ipx}}{x^2+1}\,dx = \pi e^{-p}$$

when $p \geq 0$

Similar considerations using



gives $\mathcal{F}f(p) = \int_{-\infty}^{\infty} \frac{e^{ipx}}{x^2+1}\,dx = \pi e^{p}$

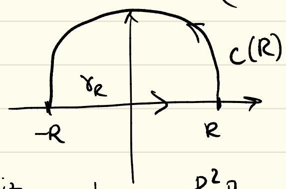
when $p<0$. So finally, $\mathcal{F}f(p) = \pi e^{-|p|}$.

We can use $\int_{-\infty}^{\infty} f(x) e^{icx}\,dx$ when $c \geq 0$ to calculate $\int_{-\infty}^{\infty} f(x)\cos(cx)\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin(cx)\,dx$ since

$$\int_{-\infty}^{\infty} f(x) e^{icx}\,dx = \int_{-\infty}^{\infty} f(x)\cos(cx)\,dx + i\int_{-\infty}^{\infty} f(x)\sin(cx)\,dx$$

Ex Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)^2}\,dx$

Solution: We use $f(z) = \dfrac{z e^{iz}}{(z^2+1)^2}$ and



$$\left| \int_{c(R)} \frac{z e^{iz}}{(z^2+1)^2}\,dz \right| \leq \frac{R^2 \pi}{(R^2-1)^2}$$

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(i, f(z)).$$ We see that $z_0 = i$ is a pole of order 2.

$$\operatorname{Res}(i, f(z)) = \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z e^{iz}}{(z-i)^2(z+i)^2} \right] = \lim_{z\to i} \frac{d}{dz}\left[\frac{z e^{iz}}{(z+i)^2} \right] =$$

$$= \lim_{z\to i} \frac{(e^{iz} + z i e^{iz})(z+i)^2 - 2(z+i) z e^{iz}}{(z+i)^4} = \lim_{z\to i} \frac{e^{iz}\left((1+iz)(z+i) - 2z\right)}{(z+i)^3} =$$

$$= \frac{e^{-1}(0-2i)}{(2i)^3} = -\frac{1}{e}\frac{2i}{(2i)^3} = \frac{1}{4e}$$

Therefore, $$\int_{\gamma_R} f(z)\,dz = \frac{2\pi}{4e} i = i\frac{\pi}{2e}$$

Let $R \to \infty$ and we conclude

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)^2}\,dx = \frac{\pi}{2e}$$

These examples makes it clear that we surely can can calculate some difficult integrals using residue calculus. Also it is clear that the choice of the contour is important. This choice is often the biggest obstacle and requires luck and cleverness.

<u>Ex</u> Calculate

$$I(a) = \int_0^\infty \frac{x^{a-1}}{x+1}\,dx \quad \text{for } 0<a<1$$

<u>Solution</u>: Here we choose $f(z) = \dfrac{z^{a-1}}{z+1}$

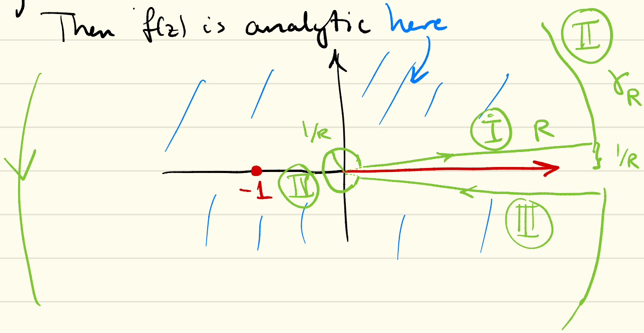
There are two "issues": z^{a-1} define on $\mathbb{C}\setminus\{\text{branch cut}\}$

$z_0 = -1$ pole

Lets choose

(not principal branch) $\Rightarrow$ $z^{a-1} := e^{(a-1)(\ln|z| + i\theta)}$ where $\theta \in [0, 2\pi)$

Then $f(z)$ is analytic here



When $R \to \infty$ $\text{I} \to \int_0^\infty \frac{x^{a-1}}{x+1}\,dx = I(a)$

$$\text{III} \to -e^{2\pi(a-1)i} \int_0^\infty \frac{x^{a-1}}{x+1}\,dx = -e^{2\pi a i} I(a)$$

$$|\text{II}| \le 2\pi R \frac{R^{a-1}}{R-1} = \frac{2\pi R^a}{R-1} \to 0$$

$$|\text{IV}| \le 2\pi \left(\frac{1}{R}\right) \frac{R^{1-a}}{2^{-1}} = \frac{4\pi}{R^a} \to 0$$

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(-1, f(z))$$

$$\operatorname{Res}(-1, f(z)) = \lim_{z \to -1} (z+1) f(z) = e^{(a-1) i\pi}$$

So $\left(1 - e^{2\pi a i}\right) I(a) = 2\pi i\, e^{i\pi(a-1)} = -2\pi i\, e^{i\pi a}$

$$e^{\pi a i}\left(e^{-\pi a i} - e^{\pi a i}\right) I(a) = -2\pi i\, e^{i\pi a}$$

$$-2i \sin(a\pi)\; I(a) = \pi(-2i)$$

$$\Rightarrow I(a) = \int_0^\infty \frac{x^{a-1}}{x+1}\,dx = \frac{\pi}{\sin(a\pi)}$$

when $0 < a < 1$.