Topology and geometry in the complex plane.

We have already seen that we can measure distance between z and the origin 0 using $|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$. (This is a so called norm)
Using the norm we can use a standard trick to define a metric (which measures distance between any z and w) as

$$d(z,w) = |z-w|.$$

Theorem 2 The function $d: \mathbb{C}\times\mathbb{C} \to [0,\infty)$ satisfies

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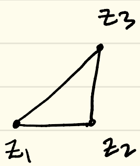
(i) $d(z_1,z_2) = d(z_2,z_1)$
(ii) $d(z_1,z_2) = 0 \iff z_1 = z_2$
(iii) $d(z_1,z_3) \le d(z_1,z_2) + d(z_2,z_3)$

(That is, d is a metric on $\mathbb{C}$)



We will leave (i) and (ii) as exercises and now work to verify (iii) (the triangle inequality)
The verification goes through a series of observations about $|z| = \sqrt{z\bar{z}}$

- $|z_1 z_2| = |z_1||z_2|$
- $|\mathrm{Re}(z)| \le |z|$ and $|\mathrm{Im}(z)| \le |z|$

($|z| = \sqrt{x^2+y^2} \ge \sqrt{x^2} = |x| = |\mathrm{Re}(z)|$ for example)

Now we investigate

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} =$$
$$= |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2 \le |z|^2 + 2|z\bar{w}| + |w|^2 =$$
$$= |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$$

$\Longrightarrow$ $|z+w| \le |z| + |w|$ by taking square roots of real numbers

This is a version of (iii) since

$$d(z_1,z_3) = |z_1 - z_3| = |z_1 - z_2 + z_2 - z_3| \le$$
$$\le |z_1 - z_2| + |z_2 - z_3| = d(z_1,z_2) + d(z_2,z_3)$$

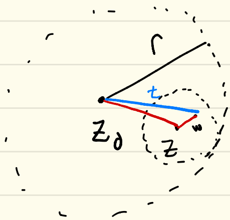
Open sets in $\mathbb{C}$

The open disk with center z_0 and radius $r>0$ is of the form $\Delta(z_0,r) = \{z \in \mathbb{C} \,;\, |z-z_0| < r\}$

A set $U \subseteq \mathbb{C}$ is open if for every $z \in U$ there is an r such that $\Delta(z,r) \subseteq U$.

Exercise: The open disk $\Delta(z_0,r)$ is an open set

Hint:



Is $t<r$?
Why ?

Other open sets are $\mathbb{C}$ and $\emptyset$ (the empty set)
(It might feel strange that the empty set is counted as an open set but if you doubt it find one point in $\emptyset$ which doesn't satisfy the defining property for an open set. You can't since $\emptyset$ has no elements)

Also, you can build new open sets by taking unions of open sets. Namely, let $U_i \subset \mathbb{C}$ be open sets for $i \in I$ (a set of indices). Then $\bigcup_{i \in I} U_i$ is open.

Finally, if U and V are open then $U \cap V$ is open. (Warning: Only finite intersections work in general)

Closed sets

A set U is called closed if it's complement $U^c = \mathbb{C} \setminus U = \{ z \in \mathbb{C} ; z \notin U \}$ is open

Ex $U = \{ z \in \mathbb{C} ; |z| \geq 1 \}$ is closed since $\mathbb{C} \setminus U = \{ z \in \mathbb{C} ; |z| < 1 \}$ is open

The set $\overline{\Delta(z_0, r)} = \{ z \in \mathbb{C} ; |z - z_0| \leq r \}$ is called the closed disk with radius r and center z_0.
(It is a closed set since $\{ z \in \mathbb{C} ; |z - z_0| > r \}$ is open. Why? This will be on one exercise sheet)

Interior points, exterior points and boundary points

Let $U \subseteq \mathbb{C}$ be a set. We say that z is an interior point of U if $\exists r>0$ such that $\Delta(z,r) \subseteq U$. We say that z is an exterior point of U if $\exists r>0$ such that $\Delta(z,r) \subseteq U^c = \mathbb{C} \setminus U$. If z is neither an interior point nor exterior point of U it is a boundary point of U. More precisely, z is a boundary point of U if for all $r>0$ we have

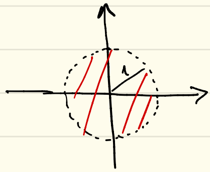
$$\Delta(z,r) \cap U \neq \emptyset \quad \text{and} \quad \Delta(z,r) \cap (\mathbb{C} \setminus U) \neq \emptyset.$$

We use the following rotation

$$\begin{aligned} \operatorname{int}(U) &= \{ z \in \mathbb{C} ;\ z \text{ is an interior point of } U \} \\ \operatorname{ext}(U) &= \{ z \in \mathbb{C} ;\ z \text{ is an exterior point of } U \} \\ \partial U &= \{ z \in \mathbb{C} ;\ z \text{ is an boundary point of } U \} \end{aligned}$$

<u>Notice</u>:
- $\mathbb{C} = \operatorname{int}(U) \cup \operatorname{ext}(U) \cup \partial U$ for any set $U \subseteq \mathbb{C}$.
- $U = \operatorname{int}(U) \iff U$ is open
- $\operatorname{ext}(U)$ and $\operatorname{int}(U)$ are always open
- $\overline{U} = U \cup \partial U$ is always a closed set ⌈This set is called the closure of U and is the smallest closed set containing U⌋

Ex $\Delta(0,1) = \{ z \in \mathbb{C} ;\ |z| < 1 \}$



$\operatorname{int}(\Delta(0,1)) = \Delta(0,1)$

$\operatorname{ext}(\Delta(0,1)) = \{ z \in \mathbb{C} ;\ |z| > 1 \}$

$\partial \Delta(0,1) = \{ z \in \mathbb{C} ;\ |z| = 1 \}$

$\overline{\Delta(0,1)} = \operatorname{int}(\Delta(0,1)) \cup \partial\Delta(0,1) = \{ z \in \mathbb{C} ;\ |z| \leq 1 \}$

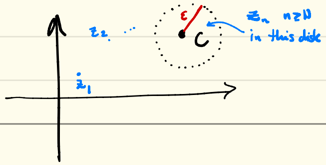
Sequences of complex numbers

Limits of sequences in $\mathbb{C}$

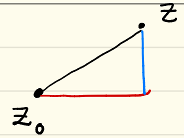
Let $(z_n)_{n=1}^{\infty}$ be a sequence where $z_n \in \mathbb{C}$.
We say $(z_n)_{n=1}^{\infty}$ has the limit $c \in \mathbb{C}$ and write $\lim_{n\to\infty} z_n = c$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$z_n \in \Delta(c, \varepsilon) \text{ whenever } n \geq N.$$

[$\forall \varepsilon > 0\ \exists N$ such that $n \geq N \Rightarrow |z_n - c| < \varepsilon$]



An observation



We see that

$$|\mathrm{Re}(z-z_0)| \le |z-z_0| \le |\mathrm{Re}(z-z_0)| + |\mathrm{Im}(z-z_0)|$$

and

$$|\mathrm{Im}(z-z_0)| \le |z-z_0| \le |\mathrm{Re}(z-z_0)| + |\mathrm{Im}(z-z_0)|$$

3 Proposition 3: Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Let $x_n = \mathrm{Re}(z_n)$ and $y_n = \mathrm{Im}(z_n)$. The following are equivalent:

(i) $\lim_{n\to\infty} z_n = c$

(ii) $\lim_{n\to\infty} x_n = \mathrm{Re}(c)$ and $\lim_{n\to\infty} y_n = \mathrm{Im}(c)$

(These are sequences of real numbers)

4 Proposition 4: Assume $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Then $\lim_{n\to\infty} c z_n = cz$, $\lim_{n\to\infty} \overline{z_n} = \overline{z}$, $\lim_{n\to\infty} |z_n| = |z|$, $\lim_{n\to\infty} z_n + w_n = z + w$, $\lim_{n\to\infty} z_n w_n = zw$, and if $w \neq 0$ $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$.

⌈ Some w_n can be zero and therefore $\frac{z_n}{w_n}$ is undefined for these. If $w \neq 0$ then only finitely many w_n are zero and are ignored ⌋

Ex Let $(z_n)_{n\geq 1}^{\infty} = \left(\frac{i^n}{n}\right)_{n\geq 1}^{\infty} = \left(i, -\frac{1}{2}, -\frac{i}{3}, \frac{1}{4}, \dots\right)$

Calculate $\lim_{n\to\infty} z_n$.

We use $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{(-1)^n}{2n} = 0$

and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{(-1)^n}{2n+1} = 0$. Therefore

$\lim_{n\to\infty} z_n = 0$.

Continuous complex-valued functions

Let $A \subseteq \mathbb{C}$ and $f: A \to \mathbb{C}$. Since $\mathbb{C}$ is $\mathbb{R}^2$ equipped with a multiplication we can write $f(z) = u(z) + i\,v(z)$ where $u: A \to \mathbb{R}$ and $v: A \to \mathbb{R}$.

Let $a \in A$ and $c \in \mathbb{C}$. We say that the limit of f at a is c and write $\lim_{z\to a} f(z) = c$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $(0 < |z-a| < \delta$ and $z \in A)$ then $|f(z) - c| < \varepsilon$

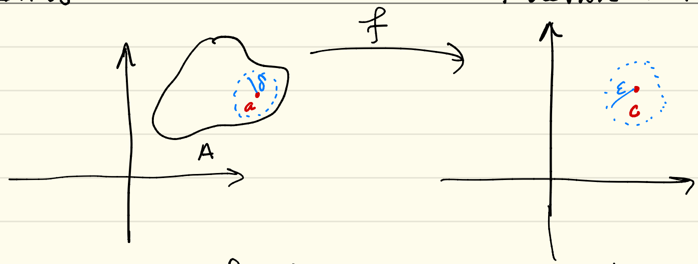
You can use what you learn in Differential and Integral Calculus 2

$$\lim_{z\to a} u(z) = \mathrm{Re}(c) \;\&\; \lim_{z\to a} v(z) = \mathrm{Im}(c)$$

$$\Longleftrightarrow$$

$$\lim_{z\to a} f(z) = c$$

Illustration of limit of a function $f : A \to \mathbb{C}$



For $\lim_{z \to a} f(z) = c$ to be true

you should be able to:

1. For any choice $\varepsilon > 0$ find
2. $\delta > 0$ so that
3. if $0 < |z - a| < \delta$ (and $z \in A$) then $|f(z) - c| < \varepsilon$

Definition: A function $f: A \to \mathbb{C}$ is a continuous function at $a \in A$ if
$$\lim_{z \to a} f(z) = f(a)$$

If $f: A \to \mathbb{C}$ is continuous at every point $a \in A$ then f is continuous in A.

Complex differentiability

Definition: We say that $f: A \to \mathbb{C}$ (A open set) is complex differentiable at $z_0 \in A$ if
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

A function that is complex differentiable at z_0 is continuous at z_0.

Let us reformulate the definition of complex differentiability

5 **Proposition 5** Let $A \subseteq \mathbb{C}$ be open and $f: A \to \mathbb{C}$. Then f is complex differentiable at $z_0 \in A$ iff there exists $E: A \to \mathbb{C}$ and $c \in \mathbb{C}$ such that
$$f(z) = f(z_0) + c(z - z_0) + E(z)$$
and
$$\lim_{z \to z_0} \frac{|E(z)|}{|z - z_0|} = 0.$$

Note: $c = f'(z_0)$