Definition: A function $f: A \to \mathbb{C}$ is a continuous function at $a \in A$ if
$$\lim_{z \to a} f(z) = f(a)$$

If $f: A \to \mathbb{C}$ is continuous at every point $a \in A$ then f is continuous in A.

Complex differentiability

Definition: We say that $f: A \to \mathbb{C}$ (A open set) is complex differentiable at $z_0 \in A$ if
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

A function that is complex differentiable at z_0 is continuous at z_0.

Let us reformulate the definition of complex differentiability

5 **Proposition 5** Let $A \subseteq \mathbb{C}$ be open and $f: A \to \mathbb{C}$. Then f is complex differentiable at $z_0 \in A$ iff there exists $E: A \to \mathbb{C}$ and $c \in \mathbb{C}$ such that
$$f(z) = f(z_0) + c(z - z_0) + E(z)$$
and
$$\lim_{z \to z_0} \frac{|E(z)|}{|z - z_0|} = 0.$$

Note: $c = f'(z_0)$

Ex $f(z) = z^2$

$$\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{\cancel{(z - z_0)}(z + z_0)}{\cancel{(z - z_0)}} = 2z_0$$

So $f'(z_0) = 2z_0$ (as expected ?)

Ex $f(z) = \overline{z}$

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} = \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \quad \text{doesn't exist}$$

Why? Because $\lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$ and

$$\lim_{y \to y_0} \frac{-y + y_0}{y - y_0} = -1$$

So $f(z) = \overline{z}$ is not complex differentiable at any point in $\mathbb{C}$.

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Proposition 6 Assume that f and g are complex differentiable at z_0. Then

(i) $(cf)'(z_0) = c(f'(z_0))$ for $c \in \mathbb{C}$

(ii) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

(iii) $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

(iv) if $g(z_0) \neq 0$ then

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

For example, $(fg)'(z_0) = \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z-z_0} =$

$$= \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z)}{z-z_0} + \lim_{z\to z_0} \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z-z_0}$$

$$= f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Chain rule

<u>Chain rule</u>: Let $f: A \to \mathbb{C}$ and $g: B \to \mathbb{C}$ be functions such that $f(A) \subset B$. Assume that f is complex differentiable at z_0 and g is complex differentiable at $w_0 = f(z_0)$. Then $g \circ f$ is complex differentiable at z_0 and $(g\circ f)'(z_0) = g'(w_0) f'(z_0) = g'(f(z_0))\, f'(z_0)$

<u>Proof</u>: We have $f(z) = f(z_0) + f'(z_0)(z - z_0) + E_1(z)$ where $\lim_{z\to z_0} \frac{|E_1(z)|}{z - z_0} = 0$ and

$$g(w) = g(w_0) + g'(w_0)(w - w_0) + E_2(w)$$

where

$$\lim_{w\to w_0} \frac{|E_2(w)|}{|w - w_0|} = 0.$$

Therefore $g(f(z)) = g(f(z_0)) + g'(f(z_0))(f(z) - f(z_0)) + E_2(f(z)) =$

$$= g(f(z_0)) + g'(f(z_0))\big(f'(z_0)(z - z_0) + E_1(z)\big) + E_2(f(z))$$

$$= g(f(z_0)) + g'(f(z_0)) f'(z_0)(z - z_0) + E_3(z)$$

where $E_3(z) = g'(f(z_0)) E_1(z) + E_2(f(z))$

If $\lim_{z \to z_0} \frac{|E_3(z)|}{|z - z_0|} = 0$ we are done.

Since $\frac{|E_3(z)|}{|z-z_0|} \le \underbrace{|g'(f(z_0))| \frac{|E_1(z)|}{|z-z_0|}}_{\to 0} + \underbrace{\frac{|E_2(f(z))|}{|f(z)-f(z_0)|}}_{\to 0} \cdot \underbrace{\frac{|f(z)-f(z_0)|}{|z-z_0|}}_{\to |f'(z_0)|}$

as $z \to z_0$

we are done. $\otimes$

One easily verifies that when $f(z) = z^n$, $n = 1, 2, \ldots$ then $f'(z) = n z^{n-1}$.

<u>Ex</u> Calculate $h'(z)$ for $h(z) = \left(\frac{z^2-1}{z^2+1}\right)^{10}$

First we calculate

$$\frac{d}{dz}\left(\frac{z^2-1}{z^2+1}\right) = \frac{2z(z^2+1) - 2z(z^2-1)}{(z^2+1)^2} = \frac{4z}{(z^2+1)^2}$$

Therefore

$$h'(z) = 10\left(\frac{z^2-1}{z^2+1}\right)^9 \cdot \frac{4z}{(z^2+1)^2} = \frac{40z(z^2-1)^9}{(z^2+1)^{11}}$$

Analytic functions

Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ be complex differentiable at every point $z \in A$. Then f is called analytic.
(A synonym is holomorphic)

At the moment we don't know so many analytic functions. Basically only polynomials of z and rational functions of z. We need more examples to make this theory interesting.

Cauchy-Riemann equations

We will use the fact that $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ gives the same value if you approach z_0 parallel to the x-axis or parallel to the y-axis if f is complex differentiable at z_0.

That is

$$(*) \quad \lim_{x \to x_0} \frac{f(x+iy_0)-f(x_0+iy_0)}{x-x_0} = \lim_{y \to y_0} \frac{f(x_0+iy)-f(x_0+iy_0)}{iy-iy_0}$$

Write $\operatorname{Re} f(x+iy) = u(x,y)$ and $\operatorname{Im} f(x+iy) = v(x,y)$

Then both u and v are real-valued functions of 2 variables.

The left-hand side of $(*)$ is

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and the right-hand side is

$$\frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x}$$

Since these have to be the same we find

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \& \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$$

These are the Cauchy-Riemann equations that hold if f is complex differentiable.

We would like to use the Cauchy-Riemann equations to check for complex differentiability. However the CR-equations alone is not enough to conclude complex differentiability. We need stronger assumptions

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__Theorem 7__ Assume that $f = u + iv$ is defined in an open set $A \subseteq \mathbb{C}$ and that the partial derivatives u_x, u_y, v_x, and v_y exist everywhere in A. Also assume that these partial derivatives are continuous at $z_0 \in A$ and that the Cauchy-Riemann equations hold at z_0.

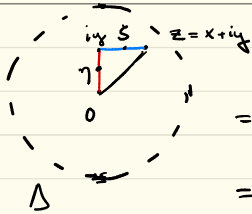
Then f is complex differentiable at z_0 (and $f'(z_0) = f_x(z_0) = -i f_y(z_0)$)

__Proof__: Assume that $z_0 = 0$ (with no loss of generality) (This assumption simplifies notation)

Let $\Delta = \Delta(0,r) \subseteq A$. Let $c = a + ib$ where $a = u_x(0) = v_y(0)$ and $b = v_x(0) = -u_y(0)$

Our goal is to show that

$$\lim_{z\to 0} \left| \frac{f(z)-f(0)}{z} - c \right| = \lim_{z\to 0} \frac{|f(z)-f(0)-cz|}{|z|} = 0$$



$$u(z)-u(0) = u(x,y)-u(0,0) =$$
$$= \underbrace{u(x,y)-u(0,y)} + \underbrace{u(0,y)-u(0,0)} =$$
$$= u_x(\xi)x + u_y(\eta)y$$ by the mean value thm

Similarly $v(z)-v(0) = v_x(\xi')x + v_y(\eta')y$ where ξ' and η' are on the same lines as ξ and η.

Note that $\max(|\xi|, |\xi'|, |\eta|, |\eta'|) \le |z|$

Now, $f(z)-f(0)-cz = u(z)-u(0)+i\big(v(z)-v(0)\big) - (a+ib)(x+iy) =$
$$= u_x(\xi)x + u_y(\eta)y - ax + by + i\big(v_x(\xi')x + v_y(\eta')y - bx - ay\big) =$$
$$= \big(u_x(\xi)-u_x(0)\big)x + \big(u_y(\eta)-u_y(0)\big)y +$$
$$i\Big(\big(v_x(\xi')-v_x(0)\big)x + \big(v_y(\eta')-v_y(0)\big)y\Big)$$

and $$\frac{|f(z)-f(0)-cz|}{|z|} \le |u_x(\xi)-u_x(0)| + |u_y(\eta)-u_y(0)| +$$
$$+ |v_x(\xi')-v_x(0)| + |v_y(\eta')-v_y(0)|$$

Since the partial derivatives are continuous at 0 and $\max(\xi, \eta, \xi', \eta') \to 0$ as $z \to 0$ we see that

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = c \qquad \otimes$$

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Thm 8 Assume $A \subseteq \mathbb{C}$ is open and also assume that $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ have continuous 1^{st} order partial derivatives. Then $f = u + iv$ is analytic if and only if the Cauchy-Riemann equations

$$u_x(z) = v_y(z) \; ; \quad u_y(z) = -v_x(z)$$

hold at each $z \in A$.

Ex Determine at which points $f(z) = 2xy + i(x^2+y^2)$ is complex differentiable. Is $f(z)$ holomorphic in some set?

Solution:

$$u(x,y) = 2xy \qquad u_x = 2y \qquad u_y = 2x$$
$$v(x,y) = x^2+y^2 \qquad v_x = 2x \qquad v_y = 2y$$

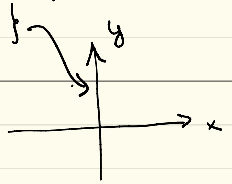
Both u and v have continuous partial derivatives so we need to check the CR-equations

$u_x = v_y$? Yes, since $u_x = 2y$ and $v_y = 2y$

What about $v_x = -u_y$? This equation holds if

$$2x = -2x \quad \text{or when } x = 0.$$

So, $f(z) = 2xy + i(x^2+y^2)$ is complex differentiable on the imaginary axis $\{(x,y) ; x = 0\}$



This is not an open set so $f(z) = 2xy + i(x^2+y^2)$ is not analytic anywhere.

Lets use the CR-equations to verify that $f(z)=e^z$ is analytic for every $z \in \mathbb{C}$.

$$f(z)=e^z = e^x(\cos y + i\sin y)$$

$$u(x,y)=e^x\cos y \qquad v(x,y)=e^x\sin y$$
$$u_x = e^x\cos y \qquad v_y = e^x\cos y$$
$$u_y = -e^x\sin y \qquad v_x = e^x\sin y$$

That is, $u_x=v_y$ & $u_y=-v_x$ and since all partial derivatives are continuous $f(z)$ is analytic for all $z\in\mathbb{C}$. Also

$$f'(z) = u_x + i v_x = e^x(\cos y + i\sin y) = e^z.$$

<u>Def</u>: An analytic function $f:\mathbb{C}\to\mathbb{C}$ is called an entire function.

⌜Polynomials and e^z are entire functions.⌟

We have $e^{it} = \cos t + i\sin t$, $t\in\mathbb{R}$, and
$e^{-it} = \cos t - i\sin t$, $t\in\mathbb{R}$
which can be rewritten

$$\cos t = \frac{e^{it}+e^{-it}}{2} \quad \& \quad \sin t = \frac{e^{it}-e^{-it}}{2i}$$

for $t\in\mathbb{R}$

We define
$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$
for all $z \in \mathbb{C}$.

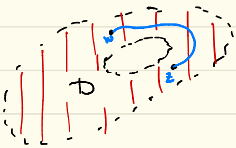
T $\cos z$ and $\sin z$ are entire functions,

Ex Verify that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$

$$\cos^2 z + \sin^2 z = \frac{(e^{iz}+e^{-iz})^2}{4} - \frac{(e^{iz}-e^{-iz})^2}{4} =$$
$$= \frac{e^{2iz}+2+e^{-2iz} - (e^{2iz}-2+e^{-2iz})}{4} = \frac{4}{4} = 1.$$

Some consequences of the Cauchy-Riemann equations

First assume that $D \subseteq \mathbb{C}$ is a non-empty, open subset where any $z, w \in D$ can be connected by a continuous curve.



Such a set is called a domain
(non-empty, open, connected (or equivalently in this case path-connected))