We define
$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$
for all $z \in \mathbb{C}$.

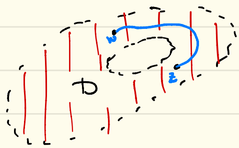
↑ $\cos z$ and $\sin z$ are entire functions.

Ex Verify that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$

$$\cos^2 z + \sin^2 z = \frac{(e^{iz}+e^{-iz})^2}{4} - \frac{(e^{iz}-e^{-iz})^2}{4} =$$
$$= \frac{e^{2iz}+2+e^{-2iz} - (e^{2iz}-2+e^{-2iz})}{4} = \frac{4}{4} = 1.$$

Some consequences of the Cauchy-Riemann equations

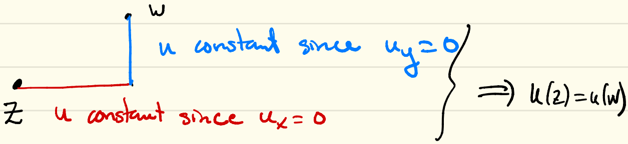
First assume that $D \subseteq \mathbb{C}$ is a non-empty, open subset where any $z, w \in D$ can be connected by a continuous curve.



Such a set is called a domain
(non-empty, open, connected (or equivalently in this case path-connected))

Assume that $u: D \to \mathbb{R}$ satisfies $u_x = u_y = 0$ for all $z \in D$. Then u is constant.

⌈



However, we need to consider

We can always find a curve of this type ⌋

9

⌈Remember D is a domain.⌋

Proposition 9 Assume $f: D \to \mathbb{C}$ is an analytic function such that $f'(z) = 0$ for all $z \in D$. Then f is constant.

Proof: Since $f'(z) = u_x + i v_x = 0$ we get $u_x = 0$ and $v_x = 0$ for all $z \in D$.
Now CR-equations gives $v_y = u_x = 0$ and $u_y = -v_x = 0$. Therefore both u and v are constant. So $f = u + iv$ is constant.

10

<u>Proposition</u> 10 Suppose $f: D \to \mathbb{C}$ is analytic (and D is a domain) and u, v or $|f|$ is constant then f is constant.

<u>Proof</u>: If u is constant then $u_x = v_y = 0$ and $u_y = -v_x = 0$. Therefore f is constant (Similar for v constant)

Assume $|f|$ is constant. We get $|f|^2 = u^2 + v^2 = c$

If $c = 0$ we are done. Assume $c > 0$

Then
$$2uu_x + 2vv_x = 0 \quad \text{and}$$
$$2uu_y + 2vv_y = 0$$

If we use the CR-equations we get
$$\begin{cases} uu_x - vu_y = 0 & (*) \\ uu_y + vu_x = 0 & (**) \end{cases}$$

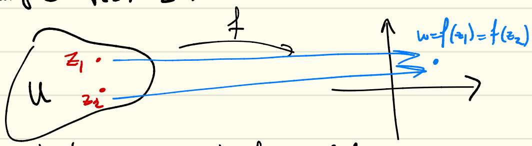
Multiply $(*)$ by u and $(**)$ by v and add to get
$$(u^2 + v^2)u_x = 0 \implies u_x = 0$$

Multiply $(*)$ by v and $(**)$ by u and subtract to get
$$(u^2 + v^2)u_y = 0 \implies u_y = 0$$

$\implies f = u + iv$ is constant. ⊠

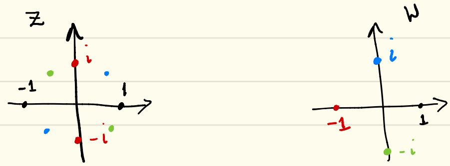
Branches of inverse functions

We have seen that many analytic functions $f: \mathcal{U} \to f(\mathcal{U}) \subseteq \mathbb{C}$ don't have single valued inverses (for example $f(z) = z^2$)



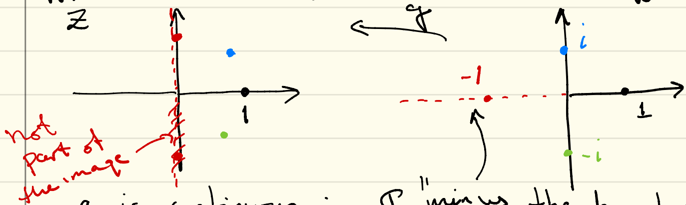
In this situation we cannot find f^{-1} such that $f \circ f^{-1}(w) = w$ and $f^{-1} \circ f(z) = z$. However, we can find many $g: f(\mathcal{U}) \to \mathcal{U}$ such that $f \circ g(w) = w$. Such a function is called a right-inverse. Most of these are very "wild". If we require that g is continuous we get a "branch of the inverse". This cannot be defined on the whole of $f(\mathcal{U})$ (unless f is injective (univalent popular in complex analysis books))

Ex: $f(z) = z^2$

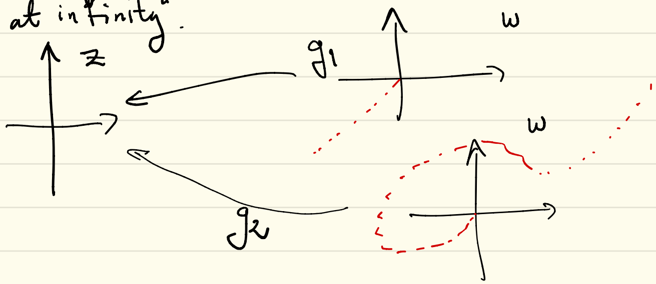


To "build" a branch start at any point in the w-plane say $w=1$ and choose one of the pre-images $z = 1$ or -1. Lets pick $g(1) = 1$. Since g is required to be continuous this determines the values of g in a small disk around 1. This process can be continued to the whole plane minus a "branch cut".

What is a branch cut ?



g is continuous in $\mathbb{C}$ "minus the branch cut"
g is nothing but the principal square root (not continuous across the branch cut). You get a different branch if you choose a different simple curve starting at the origin ("branch point") and "ending at infinity".



Had you started with the choice $g(1) = -1$ you would have gotten a different branch namely $-\sqrt{z}$.

We will now see that branches of inverses gives us many new analytic functions. They are continuous (by definition) but are they analytic ?

11

$g(D) = \{g(z);\ z \in D\}$

Theorem 11 Suppose that $f: U \to \mathbb{C}$ is an analytic function and that g is a branch of f^{-1} in a domain D. Let $z_0 \in D$ and $w_0 = g(z_0)$. If $f'(w_0) \neq 0$ then g is complex differentiable at z_0 and $g'(z_0) = 1/f'(w_0)$. (Therefore, if $f'(w) \neq 0$ when $w \in g(D)$, then g is analytic in D and $g'(z) = 1/f'(g(z))$.)

Proof: Let $w = g(z)$ for $z \in D$ (g is injective so w is well-defined)
Note that $w \neq w_0$ when $z \neq z_0$ (g injective) and $w = g(z) \to g(z_0) = w_0$ as $z \to z_0$ (g continuous)
We get

$$\frac{g(z)-g(z_0)}{z-z_0} = \frac{g(z)-g(z_0)}{f(g(z))-f(g(z_0))} = \frac{w-w_0}{f(w)-f(w_0)} \to \frac{1}{f'(w_0)}$$

and

$$g'(z_0) = \frac{1}{f'(w_0)}$$

$\blacksquare$

Ex Let $g(z)$ be a branch of the square root (That is a branch of f^{-1} where $f(w) = w^2$.)
We get $f'(w) = 2w$ and $g'(z_0) = \frac{1}{2w_0} = \frac{1}{2g(z_0)}$ when $g(z_0) = w_0 \neq 0$
For example, if $g(z)$ is the principal square root $\sqrt{z}$ we get

$$g'(z) = \frac{1}{2\sqrt{z}} \quad \text{when } z \neq 0$$

(and $\mathrm{Re}\, z > 0$ if $\mathrm{Im}\, z = 0$)

We could also use $f(g(z)) = g(z)^2 = z$ and differentiate $2g(z)g'(z) = 1$ (we need to know $g'(z)$ exists) and $g'(z) = \frac{1}{2g(z)}$ if $g(z) \neq 0$

Ex Let $g(z)$ be a branch of the logarithm. Then $e^{g(z)} = z$ and

$$g'(z)e^{g(z)} = 1 \text{ or } g'(z) = \frac{1}{e^{g(z)}} = \frac{1}{z}$$

So, for example, $\frac{d}{dz} \operatorname{Log}(z) = \frac{1}{z}$.

It is reasonable to wonder if it is possible to invert some analytic function f near $f(z_0)$ even though $f'(z_0) = 0$. It turns out that the answer is no. It is interesting to note that the situation for $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is different. Lets discuss this without too much detail.

Say $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is given as $\tilde{f}(x,y) = (u(x,y), v(x,y))^T$

We know from Differential and Integral Calculus 2 that $u: \mathbb{R}^2 \to \mathbb{R}$ is (in the real sense) differentiable at (x_0,y_0) if there is $A, B \in \mathbb{R}$ such that

$$u(x,y) = u(x_0,y_0) + A(x-x_0) + B(y-y_0) + O((x-x_0)^2 + (y-y_0)^2)$$

Similarly $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable at (x_0, y_0) if there is a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that

$$\tilde{f}(x,y) = \tilde{f}(x_0,y_0) + \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + O((x-x_0)^2 + (y-y_0)^2)$$

If $\tilde{f}$ is real differentiable in an open set containing (x_0,y_0) then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix} = d\tilde{f}(x_0,y_0)$$

If we view $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ as $f: \mathbb{C} \to \mathbb{C}$ using $f = u + iv$ and we see that real differentiability is not enough to imply complex differentiability. The CR-equations imply that

$$df(x_0,y_0) = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

⌈Another way of saying this is that df has to be complex linear and not only real linear⌋

The Inverse Function Theorem says $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is invertible near points where $\det d\tilde{f} \neq 0$. If $f: \mathbb{C} \to \mathbb{C}$ is analytic then $\det df = |f'(z)|^2 \geq 0$. It is possible that $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is invertible near points where $\det d\tilde{f} = 0$ but for analytic functions this is not the case. So for analytic functions they are invertible near points $w = f(z)$ exactly when $f'(z) \neq 0$.