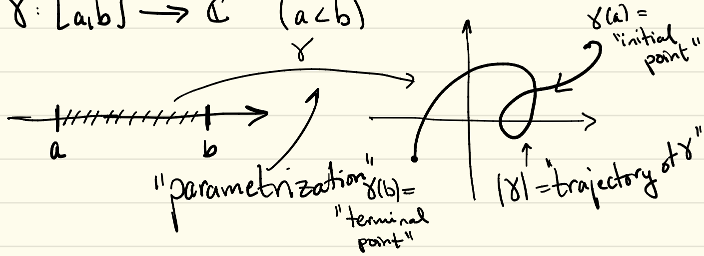
Complex integration

We want to define integrals of complex-valued functions along paths in the complex plane.

A path γ in the complex plane is a continuous map $\gamma : [a,b] \to \mathbb{C}$ $(a<b)$



Smooth paths

A path is called smooth if

$$\dot{\gamma}(t) = \lim \frac{\gamma(t+h)-\gamma(t)}{h} \quad \text{exists for all}$$

$t \in [a,b]$ and $\dot{\gamma}(t) \neq 0$.

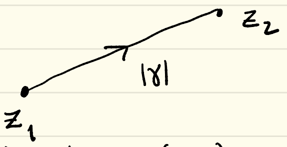
(One-sided derivatives at a and b)

A path is piecewise smooth if it is made out of several smooth paths.

A path is closed if $\gamma(a) = \gamma(b)$ and is simple if $\gamma(t) \neq \gamma(s)$ when $t \neq s$ and $a < t < b$.

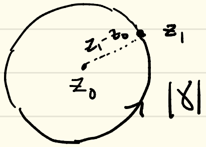
Some examples of paths

Line segments



$$\gamma : [0,1] \to \mathbb{C} \qquad \gamma(t) = t z_2 + (1-t) z_1$$

Circle



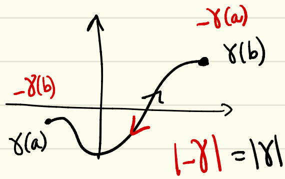
$$\gamma(t) = z_0 + (z_1 - z_0) e^{it}$$

$0 \le t \le 2\pi$

Reverse paths

Let $\gamma : [a,b] \to \mathbb{C}$ be a path. Then the reverse path $(-\gamma) : [a,b] \to \mathbb{C}$ is $-\gamma(t) = \gamma(b+a-t)$

Ex

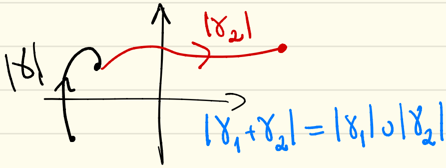


Path sum

Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be two paths with $\gamma_1(b_1) = \gamma_2(a_2)$. Then the path sum is
$\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$ given by

$$[\gamma_1 + \gamma_2](t) = \begin{cases} \gamma_1(t) & \text{if } a_1 \le t \le b_1 \\ \gamma_2(t - b_1 + a_2) & \text{if } b_1 \le t \le b_1 + b_2 - a_2 \end{cases}$$

Ex



Let $f : [a, b] \to \mathbb{C}$ be continuous and write $f(x) = u(x) + i\,v(x)$. Then

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i \int_a^b v(x)\,dx$$

The Fundamental Theorem of Calculus gives

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F(x) = U(x) + i\,V(x)$$

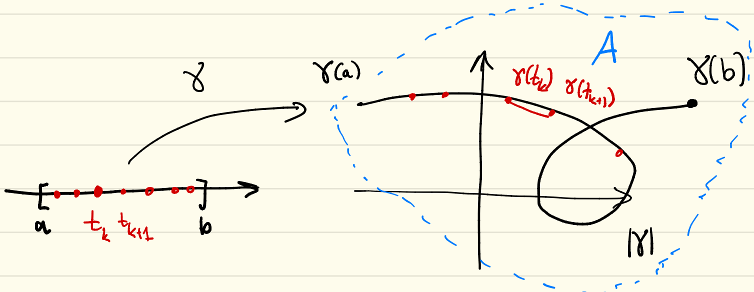
and $U'(x) = u(x)$ and $V'(x) = v(x)$.

Let $\gamma : [a,b] \to \mathbb{C}$ be a path. Let $f : A \to \mathbb{C}$ be continuous and $|\gamma| \subset A$. Then we define the contour integral of f along γ as

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\dot{\gamma}(t)\,dt$$

and the integral of f along γ with respect to arclength as

$$\int_\gamma f(z)\,|dz| := \int_a^b f(\gamma(t))\,|\dot{\gamma}(t)|\,dt$$



$$\gamma(t_{k+1}) - \gamma(t_k) \approx \gamma'(t_k)\,(t_{k+1} - t_k)$$
$$|\gamma(t_{k+1}) - \gamma(t_k)| \approx |\gamma'(t_k)|\,(t_{k+1} - t_k)$$

$$\Longrightarrow \int_\gamma f(z)\,dz \approx \sum_k f(\gamma(t_k))\,(\gamma(t_{k+1}) - \gamma(t_k))$$
$$\int_\gamma f(z)\,|dz| \approx \sum_k f(\gamma(t_k))\,|\gamma(t_{k+1}) - \gamma(t_k)|$$

Properties of ordinary Riemann integrals immediately carry over to these integrals

$$\int_\gamma f(z) + c g(z)\, dz = \int_\gamma f(z) dz + c \int_\gamma g(z) dz$$

$$\int_\gamma f(z) + c g(z)\, |dz| = \int_\gamma f(z) |dz| + c \int_\gamma g(z) |dz|$$

We also have

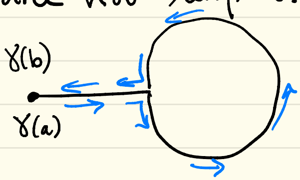
$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, |dz| \quad \text{since}$$

$$\left| \sum_k f(\gamma(t_k))(\gamma(t_{k+1}) - \gamma(t_k)) \right| \leq \sum_k |f(\gamma(t_k))|\, |\gamma(t_{k+1}) - \gamma(t_k)|$$

Def: The length of γ is

$$\ell(\gamma) = \int_\gamma 1\, |dz|$$

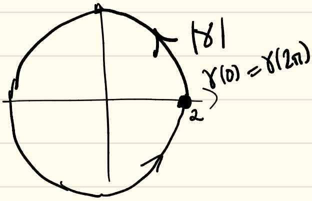
Notice not length of $|\gamma|$! Why?



<u>Ex</u> Evaluate $\int_\gamma \frac{1}{z}\,dz$ and $\int_\gamma \frac{1}{z^2}\,|dz|$

where $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$.

<u>Solution:</u>



$\dot{\gamma}(t) = 2ie^{it}$

$|\dot{\gamma}(t)| = 2$

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{2e^{it}} \cdot 2ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

$$\int_\gamma \frac{1}{z^2}\,|dz| = \int_0^{2\pi} \frac{1}{4e^{2it}}\,2\,dt = \frac{1}{2}\int_0^{2\pi} e^{-2it}\,dt =$$

$$= \frac{1}{2}\int_0^{2\pi} \cos(2t) - i\sin(2t)\,dt = 0$$

⌜In fact, we could also use $\frac{d}{dt}e^{-2it} = -2ie^{-2it}$⌟

We also have the following properties

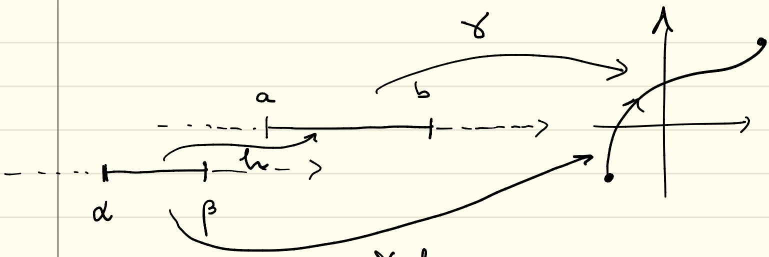
$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$$

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

$$\int_{\gamma_1+\gamma_2} f(z)\,|dz| = \int_{\gamma_1} f(z)\,|dz| + \int_{\gamma_2} f(z)\,|dz|$$

<u>BUT</u> $$\int_{-\gamma} f(z)\,|dz| = \int_\gamma f(z)\,|dz|$$

Reparametrization of curves and integrals



$$\eta = \gamma \circ h$$

We say that η is a reparametrization of γ with change of parameter h if h is continuous and strictly increasing with $h(\alpha) = a$ and $h(\beta) = b$. We say that h is a smooth change of parameter if h is smooth as a function. (Piecewise smooth in the same way)

If we use the chain rule we see that

$$\int_\eta f(z)\,dz = \int_\gamma f(z)\,dz$$

and

$$\int_\eta f(z)\,|dz| = \int_\gamma f(z)\,|dz|$$

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Proposition 12 If $|f(z)| \le M$ on $|\gamma|$ then

$$\left| \int_\gamma f(z)\,dz \right| \le M\,\ell(\gamma)$$

Proof:

$$\left| \int_\gamma f(z)\,dz \right| \le \int_\gamma |f(z)|\,|dz| \le M \int_\gamma |dz| = M\,\ell(\gamma)$$

⊗

Ex Let $\gamma(t) = 2e^{it}$ when $-\frac{\pi}{6} \le t \le \frac{\pi}{6}$.

Estimate $\left| \int_\gamma \frac{1}{z^3+1}\,dz \right|$ from above.

Solution: Notice that $|\gamma(t)| \equiv 2$. Therefore

$$|z^3+1| \ge |z^3| - 1 = 8 - 1 = 7 \quad \text{on } |\gamma|.$$

Hence $\left| \frac{1}{z^3+1} \right| \le \frac{1}{7}$ on $|\gamma|$ and

$$\left| \int_\gamma \frac{1}{z^3+1}\,dz \right| \le \frac{1}{7} \cdot \frac{2\pi}{6} \cdot 2 = \frac{2\pi}{21}$$

Primitive functions

Assume that $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$. A function $F : U \to \mathbb{C}$ is a primitive function of f if

F is **analytic** and $F'(z) = f(z)$ for $z \in U$.

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Theorem 13 Assume that f is continuous on an open set U and that F is a primitive function of f in U. If $\gamma:[a,b]\to U$ is a piecewise smooth path, then

$$\int_\gamma f(z)\,dz = \Big[F(z)\Big]_{\gamma(a)}^{\gamma(b)} = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Assume $\gamma:[a,b]\to U$ is smooth
(The case γ piecewise smooth follows easily from this.)

Write $\gamma(t) = x(t)+iy(t)$ and put

$$G(t) = F(\gamma(t)) = U(x(t),y(t)) + iV(x(t),y(t)).$$

Then

$$G'(t) = \overset{1}{U_x\cdot x'(t)} + \overset{2}{iV_x\cdot x'(t)} + \overset{3}{U_y\cdot y'(t)} + \overset{4}{iV_y\cdot y'(t)}$$

by the chain rule. The CR-equations give

$$\begin{aligned}G'(t) &= \overset{1}{U_x\cdot x'(t)} + \overset{4}{iU_x\cdot y'(t)} + \overset{2}{iV_x\cdot x'(t)} - \overset{3}{V_x\cdot y'(t)} \\ &= U_x\cdot(x'(t)+iy'(t)) + iV_x\cdot(x'(t)+iy'(t)) = \\ &= (U_x+iV_x)\cdot\gamma'(t) = F'(\gamma(t))\gamma'(t) = \\ &= f(\gamma(t))\cdot\gamma'(t).\end{aligned}$$

Therefore

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b G'(t)\,dt = G(b)-G(a) =$$

$$= F(\gamma(b)) - F(\gamma(a)) = \Big[F(z)\Big]_{\gamma(a)}^{\gamma(b)} \qquad \otimes$$

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Corollary 14 If $f : U \to \mathbb{C}$ is a continuous function on an open set $U \subseteq \mathbb{C}$ with a primitive (analytic) function $F : U \to \mathbb{C}$ then

$$\int_\gamma f(z)\,dz = 0$$ for every closed, piecewise smooth path γ in U.

Proof

$$\int_\gamma f(z)\,dz = \Big[F(z)\Big]_{\gamma(a)}^{\gamma(b)} = 0$$

$\gamma(a) = \gamma(b)$

$\otimes$

Ex Calculate $\int_\gamma z^2\,dz$ when

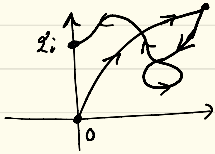
$$\gamma(t) = t^2 + it, \qquad 0 \le t \le 1$$

Solution: Since $F(z) = \frac{z^3}{3}$ is a primitive function for z^2 we get

$$\int_\gamma z^2\,dz = \left[\frac{z^3}{3}\right]_{\gamma(0)}^{\gamma(1)} = \left[\frac{z^3}{3}\right]_0^{1+i} =$$

$$= \frac{(1+i)^3}{3} = \frac{1+3i+3i^2+i^3}{3} = \frac{-2+2i}{3}$$

Ex Evaluate $\int_\gamma z\sin z\,dz$ when $|\gamma|$ is the curve



Solution: We can use partial integration to find a primitive function since we have

$$\frac{d}{dz}f(z)g(z) = f'(z)g(z) + f(z)g'(z)$$

for analytic functions.

$$\int_\gamma z\sin z\,dz = \left[-z\cos z\right]_0^{2i} + \int_\gamma \cos z\,dz =$$

$$= -2i\cos 2i + \left[\sin z\right]_0^{2i} =$$

$$= -2i\left(\frac{e^{2i^2}+e^{-2i^2}}{2}\right) + \sin 2i =$$

$$= -i\left(e^{-2}+e^{2}\right) + \frac{e^{2i^2}-e^{-2i^2}}{2i} =$$

$$= -i\left(e^{-2}+e^{2}+\frac{1}{2}e^{-2}-\frac{1}{2}e^{2}\right) =$$

$$= -i\left(\frac{1}{2}e^{2}+\frac{3}{2}e^{-2}\right)$$

Ex The analytic function $f(z) = \frac{1}{z}$ defined in $\mathbb{C}\setminus\{0\}$ has no primitive function!

This must be true since $\int_\gamma \frac{1}{z}\,dz = 2\pi i$

when γ is the unit circle oriented counterclockwise.

This might seem strange since we already know that for any branch $g(z)$ of the inverse of e^z we have $g'(z) = \frac{1}{z}$. However, it is impossible to define $g(z)$ on $\mathbb{C} \setminus \{0\}$! For example, $Log(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ (but not $\mathbb{C} \setminus \{0\}$)

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A look ahead

- We know if $f(z)$ has an analytic primitive function $F(z)$ then $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth γ.
- We will now start proving Cauchy's Theorem which says that if $f(z)$ is analytic then $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth γ.
- We also aim to prove the converse namely if $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth γ then $f(z)$ is analytic (Morera's Theorem)

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We begin by proving Cauchy's Theorem for rectangles. We will need a version of Cantor's Theorem.

Special case of Cantor's Theorem

[Let R_i be a decreasing sequence of closed rectangles. Then $\bigcap_{i=1}^{\infty} R_i \neq \emptyset$
(If $\operatorname{diam} R_i \to 0$ then $\bigcap_{i=1}^{\infty} R_i = \{z\}$)]