This might seem strange since we already know that for any branch $g(z)$ of the inverse of e^z we have $g'(z) = \frac{1}{z}$. However, it is impossible to define $g(z)$ on $\mathbb{C} \setminus \{0\}$! For example, $Log(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ (but not $\mathbb{C} \setminus \{0\}$)

A look ahead

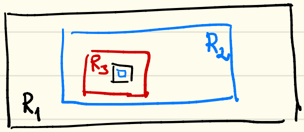
- We know if $f(z)$ has an analytic primitive function $F(z)$ then $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth γ.
- We will now start proving Cauchy's Theorem which says that if $f(z)$ is analytic then $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth γ.
  (This version true in disks)
- We also aim to prove the converse namely if $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth γ then $f(z)$ is analytic (Morera's Theorem)

We begin by proving Cauchy's Theorem for rectangles. We will need a version of Cantor's Theorem.

Special case of Cantor's Theorem

Let R_i be a decreasing sequence of closed rectangles. Then $\bigcap_{i=1}^{\infty} R_i \neq \emptyset$
(If $\operatorname{diam} R_i \to 0$ then $\bigcap_{i=1}^{\infty} R_i = \{z\}$)

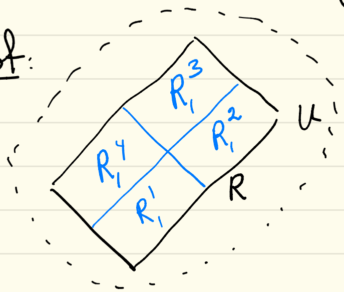
Illustration of Cantor's Theorem



15 **Lemma 15 (Cauchy's Theorem for rectangles)**

If f is analytic in an open set U then $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle R in U.

Proof:



We see that

$$\int_{\partial R} f(z)\,dz = \int_{\partial R_1^1} + \int_{\partial R_1^2} + \int_{\partial R_1^3} + \int_{\partial R_1^4}$$

(with obvious abbreviation)

Put $I = \int_{\partial R} f(z)\,dz$ and $I_1^j = \int_{\partial R_1^j} f(z)\,dz$.

The triangle inequality gives $|I| \le |I_1^1| + |I_1^2| + |I_1^3| + |I_1^4|$.

Let $|I_1| = \max\{|I_1^1|, |I_1^2|, |I_1^3|, |I_1^4|\}$. We see that

$|I| \le 4\,|I_1|$ (otherwise we get a contradiction)

Keep dividing and (with (hopefully?) obvious notation) we get

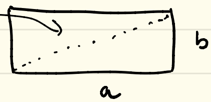
$$|I| \le 4^n |I_n| \quad \text{for } n = 1, 2, 3, \ldots$$

Let R_n be the rectangle corresponding to $|I_n|$.

⌈ diameter = "maximal distance between two points in the set" ⌋

By Cantor's Theorem $\bigcap_{n=1}^{\infty} R_n \neq \emptyset$ and therefore $\exists\, z_0$ such $z_0 \in R_n$ for $n = 1, 2, 3, \ldots$

Note that the diameter of the rectangle R is $\sqrt{a^2+b^2}$ if R is



Also the perimeter of R is $2a+2b$

Let $d = \sqrt{a^2+b^2}$ and $L = 2a+2b$. Note that the diameter of $R_n = d_n = \frac{1}{2^n} d$ and the perimeter of $R_n = L_n = \frac{1}{2^n} L$

Since f is analytic we have (for z_0 defined above)

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + E(z) \text{ where } \lim_{z \to z_0} \frac{|E(z)|}{|z-z_0|} = 0.$$

This means that for $\varepsilon > 0$ there exists $\delta > 0$ such that $|z-z_0| < \delta$ then $|E(z)| < \varepsilon |z-z_0|$. Choose n so big that R_n fits in the disk $\{z \in \mathbb{C};\ |z-z_0| < \delta\}$ ($d_n = \frac{1}{2^n} d < \delta$ does the trick). Then

$$I_n = \int_{\partial R_n} f(z)\,dz = \int_{\partial R_n} f(z_0) + f'(z_0)(z-z_0) + E(z)\,dz =$$

$$= \int_{\partial R_n} E(z)\,dz \quad \left(\text{since } \frac{d}{dz} z = 1 \text{ and } \frac{d}{dz} \frac{(z-z_0)^2}{2} = (z-z_0) \right)$$

We get $|I_n| = \left| \int_{\partial R_n} E(z)\,dz \right| \leq \int_{\partial R_n} |E(z)|\,|dz| \leq \varepsilon d_n L_n$

Recall $|I| \leq 4^n |I_n| \leq 4^n \varepsilon d_n L_n = 4^n \varepsilon \frac{d}{2^n} \frac{L}{2^n} = \varepsilon d L$

Since $\varepsilon > 0$ is arbitrary we get $|I| = 0$ and

$$I = \int_{\partial R} f(z)\,dz = 0 \qquad \otimes$$

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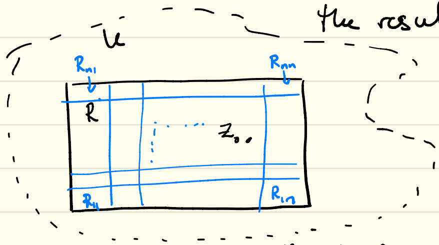
We will weaken the assumptions a little in the previous result. It will turn out to be useful in the future. (when we prove Cauchy's Integral Formula)

Lemma 16 If a function f is continuous in an open set U and analytic in $U \setminus \{z_0\}$ for some point z_0 of U, then $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle R in U.

Proof: Assume $z_0 \in R$ (otherwise we already know the result is true)



For an integer n divide R into n^2 congruent rectangles

We have $$\int_{\partial R} f(z)\,dz = \sum_{k=1}^{n}\sum_{l=1}^{n}\int_{\partial R_{kl}} f(z)\,dz$$

If $z_0 \notin R_{kl}$ then $\int_{\partial R_{kl}} f(z)\,dz = 0$ and if $z_0 \in R_{kl}$ then $\left|\int_{\partial R_{kl}} f(z)\,dz\right| \leq \frac{ML}{n}$ where L is the perimeter of R and $M = \max\{|f(z)|, z \in R\}$

Since z_0 is contained in at most 4 rectangles we get $$\left|\int_{\partial R} f(z)\,dz\right| \leq \frac{4ML}{n}$$ and n is arbitrary we get $$\int_{\partial R} f(z)\,dz = 0$$ ⊗

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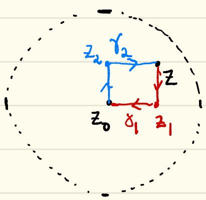
Lemma 17 Let Δ be an open disk in the complex plane, and let $f:\Delta\to\mathbb{C}$ be continuous with the property that
$$\int_{\partial R} f(z)\,dz=0$$
for every rectangle R in Δ with sides parallel to the coordinate axes. Then f has an analytic primitive in Δ. ⌈Meaning as a consequence $\int_\gamma f(z)\,dz=0$ for every closed, piecewise smooth path γ in Δ.⌋

Proof: Let z_0 be the center of Δ.



$z_0=x_0+iy_0$ $\qquad$ $z_1=x+iy_0$

$z=x+iy$ $\qquad$ $z_2=x_0+iy$

Define $$F(z)=\int_{\gamma_2} f(w)\,dw = \int_{-\gamma_1} f(w)\,dw$$
(well-defined since $\int_{\partial R} f(z)\,dz=0$)

We have
$$F(z)=\int_{y_0}^{y} f(x_0+it)\,i\,dt+\int_{x_0}^{x} f(t+iy)\,dt = $$
$$=\int_{x_0}^{x} f(t+iy_0)\,dt+\int_{y_0}^{y} f(x+it)\,i\,dt$$

Now, by the Fundamental Theorem of Calculus,
$$\frac{\partial}{\partial x}F(z)=f(x+iy)=f(z) \quad\text{and}\quad \frac{\partial}{\partial y}F(z)=if(x+iy)=if(z)$$

So F satisfies the CR-equations, has continuous partial derivatives and hence is analytic. Also $F'(z)=f(z)$

⌈ $$\left.\begin{aligned}\frac{\partial F}{\partial x}&=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=u(z)+iv(z)\\ \frac{\partial F}{\partial y}&=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=i(u(z)+iv(z))=-v(z)+iu(z)\end{aligned}\right\}\Rightarrow \begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x}&=-\frac{\partial u}{\partial y}\end{aligned}$$ ⌋

18

Theorem 18 (Cauchy's Theorem - Local Form)

Assume that Δ is an open disk in the complex plane and that f is analytic in Δ (continuous in Δ and analytic in $\Delta \setminus \{z_0\}$ enough). Then

$$\int_\gamma f(z)\,dz = 0 \text{ for every closed, piecewise}$$

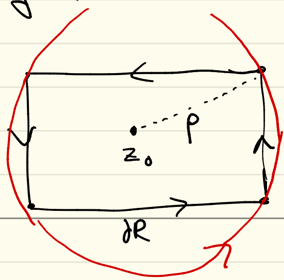
smooth path γ in Δ.

Proof: By Lemma 16 $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle in Δ and by Lemma 17 $f : \Delta \to \mathbb{C}$ has an analytic primitive function in Δ. By Corollary 14 the result follows. $\otimes$

Notice that the path must be contained in a disk. We will later remove this restriction. The following example hints at ways of doing this.

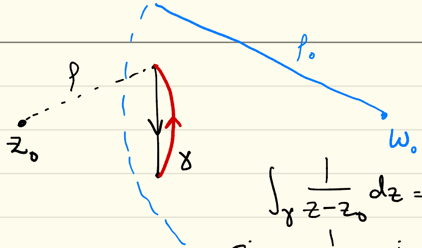
Ex Let R be a rectangle with center z_0. Show that $\int_{\partial R} \frac{1}{z-z_0}\,dz = 2\pi i$ if ∂R is oriented positively (counterclockwise)

Solution:



We know that $\int_{\gamma_{\Delta(z_0,\rho)}} \frac{1}{z-z_0}\,dz = 2\pi i$

We see



$$\int_\gamma \frac{1}{z-z_0}\,dz = 0$$

since $\frac{1}{z-z_0}$ is holomorphic in $\Delta(w_0,\rho_0)$.

Repeat for all sides of R and we get $\displaystyle\int_{\partial R} \frac{1}{z-z_0}\,dz = \int_{\partial\Delta(z_0,\rho)} \frac{1}{z-z_0}\,dz = 2\pi i$. $\otimes$

The next result will give us a way of creating analytic functions.

19 **Proposition 19** Let γ be a piecewise smooth path in the complex plane, let h be a function continuous on $|\gamma|$, and let k be a positive integer ($k=1,2,3,\ldots$). The function H defined in the open set $U=\mathbb{C}\setminus|\gamma|$ by

$$H(z) = \int_\gamma \frac{h(\zeta)}{(\zeta-z)^k}\,d\zeta$$

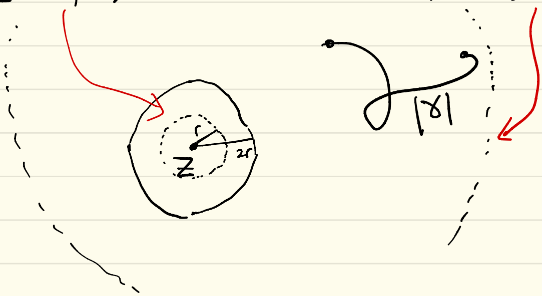
is analytic and

$$H'(z) = k\int_\gamma \frac{h(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta.$$

Proof: We want to show that, for $z \in \mathbb{C} \setminus |\gamma|$, we have

$$\left| \frac{H(z+w)-H(z)}{w} - k\int_\gamma \frac{h(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right| \to 0$$

as $w \to 0$. Let $r>0$ and $s>0$ such that disks $\Delta(z, 2r)$ and $\Delta(0,s)$ satisfies $\Delta(z, 2r) \subseteq \mathbb{C}\setminus|\gamma|$ and $|\gamma| \subseteq \Delta(0,s)$



For $w \neq 0$ we have

$$\frac{H(z+w)-H(z)}{w} - k\int_\gamma \frac{h(\zeta)}{(\zeta-z)^{k+1}} d\zeta =$$

$$= \int_\gamma h(\zeta) \underbrace{\left(\frac{1}{w}\left(\frac{1}{(\zeta-z-w)^k} - \frac{1}{(\zeta-z)^k} \right) - \frac{k}{(\zeta-z)^{k+1}} \right)}_{*} d\zeta$$

Put $a = \zeta - z$.

Then $\circledast$ is

$$\frac{1}{w}\left(\frac{1}{(a-w)^k}-\frac{1}{a^k}\right)-\frac{k}{a^{k+1}}=\frac{1}{w}\left(\frac{a^k-(a-w)^k}{a^k(a-w)^k}\right)-\frac{k}{a^{k+1}}=$$

$$=\frac{1}{w}\left(\frac{a^k-\sum_{j=0}^{k}\binom{k}{j}a^{k-j}(-w)^j}{a^k(a-w)^k}\right)-\frac{k}{a^{k+1}}=\frac{\sum_{j=1}^{k}\binom{k}{j}a^{k-j}(-w)^{j-1}}{a^k(a-w)^k}-\frac{k}{a^{k+1}}=$$

$$=\frac{\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}(-w)^{j-1}-k(a-w)^k}{a^{k+1}(a-w)^k}=\frac{k a^k+\sum_{j=2}^{k}\binom{k}{j}a^{k+1-j}(-w)^{j-1}-k\sum_{j=0}^{k}\binom{k}{j}a^{k-j}(-w)^j}{a^{k+1}(a-w)^k}=$$

$$=\frac{\sum_{j=2}^{k}\binom{k}{j}a^{k+1-j}(-w)^{j-1}-k\sum_{j=1}^{k}\binom{k}{j}a^{k-j}(-w)^j}{a^{k+1}(a-w)^k}=(-w)\frac{\left(\sum_{j=2}^{k}\binom{k}{j}a^{k+1-j}(-w)^{j-2}-k\sum_{j=1}^{k}\binom{k}{j}a^{k-j}(-w)^{j-1}\right)}{a^{k+1}(a-w)^k}$$

Notice that $r<|z-\varsigma|=|a|<s$ and if $|w|<r$ we get $|a-w|\geq r$ then $|\circledast|\leq \frac{C s^{k-1}}{r^{2k-1}}|w|$

If $L=\ell(\gamma)$ and $M=\max_{z\in|\gamma|}|h(z)|$ (which both are finite since $|\gamma|$ is compact and h continuous) we see

$$\left|\frac{H(z+w)-H(z)}{w}-k\int_\gamma \frac{h(\varsigma)}{(\varsigma-z)^{k+1}}\,d\varsigma\right|\leq$$

$$\leq \frac{C s^{k-1}}{r^{2k-1}}LM|w|\to 0 \quad \text{as} \quad |w|\to 0$$

$\otimes$

Do $k=1$ as an exercise (if needed :))

We can use this result to "create" many analytic functions. However, the functions created are not always so easy to work with.

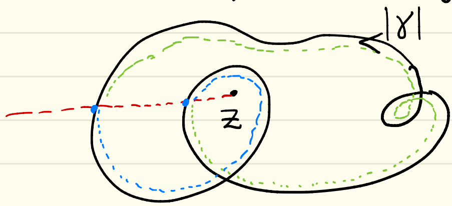
Winding Numbers

Let γ be a closed piecewise smooth path and let z be a point in $\mathbb{C}\setminus|\gamma|$.

<u>Def</u>: The winding number (or index) of γ about z is

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}.$$

<u>Properties</u>: • $n(\gamma, z)$ is always an integer.

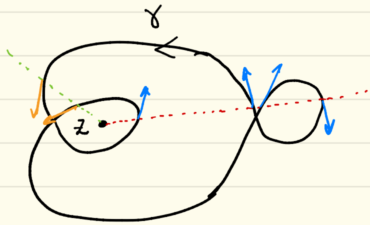


A branch of $\log(\zeta - z)$ is a primitive analytic function of $\frac{1}{\zeta - z}$

$2\pi i \; n(\gamma, z) = 2\pi i + 2\pi i$

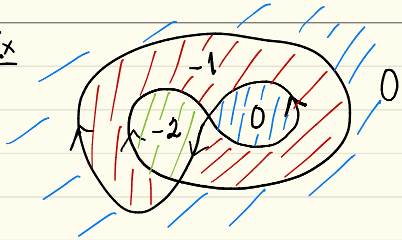
• $n(\gamma, z)$ is locally constant (since $n(\gamma, z)$ is analytic in $\mathbb{C}\setminus|\gamma|$ and hence continuous)

<u>Ex</u>:



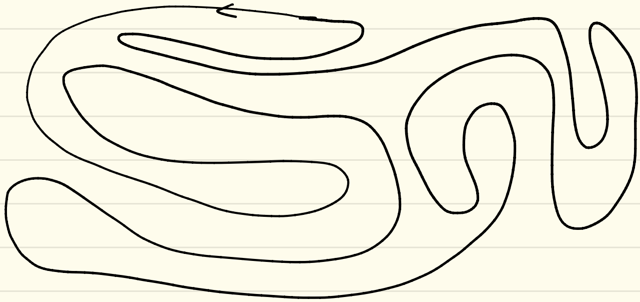
$n(\gamma, z) = 2$

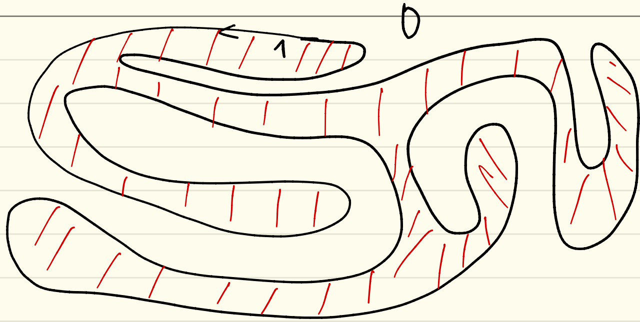
Ex



$n(\gamma, z) = 0$ always true in unbounded component

• If γ is a simple closed piecewise smooth path (simple = no self-intersection (except for $\gamma(a) = \gamma(b)$))
then $n(\gamma, z) = 1$ or $n(\gamma, z) = -1$ if z is in bounded component.
⌈ This depends on the Jordan Curve Theorem that states that a simple closed path divides the plane in two components (one bounded and one unbounded)
This seems obvious but is difficult to prove.





Terminology: We say that a simple, closed, and piecewise smooth path γ is positively oriented if $n(\gamma, z) = 1$ for z in the bounded component (Negatively oriented if $n(\gamma, z) = -1$ in the bounded component.)

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Theorem 20 (Cauchy's Integral Formula - Local Form)

Suppose f is analytic in an open disk Δ and that γ is a closed, piecewise smooth path in Δ. Then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for every $z \in \Delta \setminus |\gamma|$