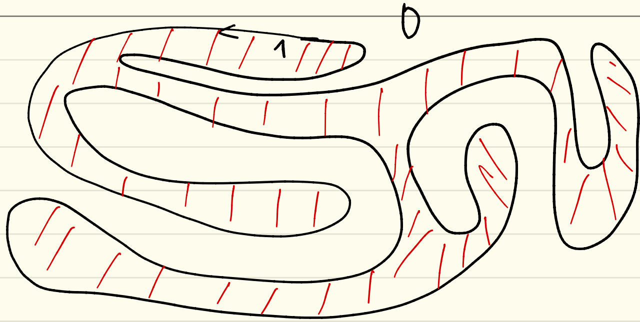

Terminology: We say that a simple, closed, and piecewise smooth path γ is <u>positively</u> <u>oriented</u> if $n(\gamma, z) = 1$ for z in the bounded component (Negatively oriented if $n(\gamma, z) = -1$ in the bounded component.)

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20 <u>Theorem 20</u> (Cauchy's Integral Formula - Local Form)

Suppose f is analytic in an open disk Δ and that γ is a closed, piecewise smooth path in Δ.

Then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for every $z \in \Delta \setminus |\gamma|$

Proof: Define, for fixed $z \in \Delta \setminus |\gamma|$,

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z. \end{cases}$$

The function g is analytic when $\zeta \neq z$ and since $\lim_{\zeta \to z} g(\zeta) = f'(z) = g(z)$

it is continuous in z. Therefore by Cauchy's Theorem (Theorem 18)

$$0 = \frac{1}{2\pi i} \int_\gamma g(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{\zeta - z} d\zeta$$

$$\Longrightarrow \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} f(z) \int_\gamma \frac{d\zeta}{\zeta - z} =$$

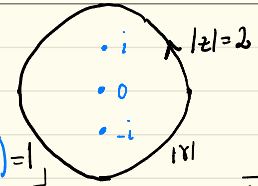
$$= f(z) \frac{2\pi i\, n(\gamma, z)}{2\pi i}$$

$$\Longrightarrow n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

$\otimes$

We will now start reaping benefits.

Ex Calculate $\int_\gamma (z^3 + z)^{-1} e^{\pi z}\, dz$ where $|\gamma| = \{|z| = 2\}$ positively oriented.



$n(\gamma, \cdot) = 1$

$$\frac{e^{\pi z}}{z^3+z} = e^{\pi z}\frac{1}{z(z^2+1)} = e^{\pi z}\frac{1}{z(z+i)(z-i)}$$

Partial fractions decomposition

$$\frac{1}{z(z+i)(z-i)} = \frac{1}{z} + \frac{-1/2}{z+i} + \frac{-1/2}{z-i}$$

Therefore
$$\int_\gamma \frac{e^{\pi z}}{z^3+z}\,dz = \int_\gamma \frac{e^{\pi z}}{z}\,dz - \frac{1}{2}\int_\gamma \frac{e^{\pi z}}{z+i}\,dz - \frac{1}{2}\int \frac{e^{\pi z}}{z-i}\,dz =$$

$$= 2\pi i\left(e^{\pi 0} - \frac{1}{2}e^{-\pi i} - \frac{1}{2}e^{\pi i}\right) = 2\pi i\left(1 + \frac{1}{2} + \frac{1}{2}\right) = 4\pi i$$

Ex Calculate $\displaystyle\int_0^{\infty} \frac{\cos t}{t^2+1}\,dt$.

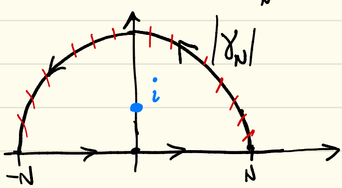
Solution: We want calculate $\displaystyle\lim_{N\to\infty}\int_0^N \frac{\cos t}{t^2+1}\,dt$

Notice that $\dfrac{\cos(-t)}{(-t)^2+1} = \dfrac{\cos t}{t^2+1}$ and $\dfrac{\sin t}{t^2+1} = -\dfrac{\sin(-t)}{(-t)^2+1}$.

Therefore
$$\lim_{N\to\infty}\int_0^N \frac{\cos t}{t^2+1}\,dt = \frac{1}{2}\lim_{N\to\infty}\int_{-N}^{N}\frac{\cos t + i\sin t}{t^2+1}\,dt =$$
$$= \frac{1}{2}\lim_{N\to\infty}\int_{-N}^{N}\frac{e^{it}}{t^2+1}\,dt.$$

We have
$$\frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)}$$ and writing $f(z) = \dfrac{e^{iz}}{z+i}$

we get $\displaystyle f(i) = \frac{1}{2\pi i}\int_{\gamma_N}\frac{f(z)}{z-i}\,dz$ where



We call the red part $|\beta|$

So $\displaystyle\int_{\gamma_N} \frac{e^{iz}}{z^2+1}\,dz = 2\pi i\,\frac{e^{i^2}}{2i} = \pi e^{-1} = \frac{\pi}{e}$. Also

$$\int_{\gamma_N} \frac{e^{iz}}{z^2+1}\,dz = \int_{-N}^{N} \frac{e^{it}}{t^2+1}\,dt + \int_{\beta} \frac{e^{iz}}{z^2+1}\,dz.$$

We get $\displaystyle\left|\int_{\beta} \frac{e^{iz}}{z^2+1}\,dz\right| \le \int_{\beta} \left|\frac{e^{iz}}{z^2+1}\right||dz| \le \int_{\beta} \frac{1}{N^2-1}\,|dz| =$

$$= \frac{\pi N}{N^2-1}$$

Therefore

$$\lim_{N\to\infty} \int_{-N}^{N} \frac{e^{it}}{t^2+1}\,dt = \frac{\pi}{e} \quad \text{and}$$

$$\int_0^{\infty} \frac{\cos t}{t^2+1}\,dt = \frac{\pi}{2e} \qquad \otimes$$

We see that Cauchy's Integral Formula gives us a way of calculating some real integrals. We will later get even more techniques for doing this. But now we turn to some important theoretical results.

21 **Theorem 21** If f is analytic in an open set U then f' is also analytic in U.

Proof: Pick a point $z \in U$ and an open disk $\Delta \subseteq U$ with center z. Then we know that

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \text{(when } \partial\Delta \text{ is positively oriented)}$$

Now using Proposition 19 we see that $f'(z)$ can be written
$$f'(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta .$$
Using the same proposition we also see that f' is analytic in Δ. Since the argument can be repeated for any $z \in \mathcal{U}$ we find that f' is holomorphic in $\mathcal{U}$. $\otimes$

Another thing that we can conclude is that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ are continuous and therefore $f \in C^1(\mathcal{U})$.

22 <u>Corollary 22</u>: If f is analytic in an open set $\mathcal{U}$ then $f^{(k)}$ is analytic in $\mathcal{U}$ for $k = 1, 2, 3, \ldots$. Moreover $f \in C^\infty(\mathcal{U})$.

23 <u>Corollary 23</u> (Morera's Theorem)
<u>If</u> f is a continuous function in an open set $\mathcal{U}$ such that $\int_{\partial R} f(z)\, dz = 0$ for every rectangle with sides parallell to the coordinate axes then f is analytic in $\mathcal{U}$.

<u>Proof</u>: By Lemma 17 f has an analytic primitive function F in $\mathcal{U}$. Since $f(z) = F'(z)$ then f is analytic in $\mathcal{U}$ $\otimes$

24 **Proposition 24** Let f be a continuous function in an open set U and assume that f is analytic in $U \setminus \{z_0\}$ for some point $z_0 \in U$. Then f is analytic in U.

Proof Combine Lemma 16 and Morera's Theorem. $\otimes$

In fact f can be assumed to be only bounded in U and analytic in $U \setminus \{z_0\}$ for the conclusion to hold. This is the Riemann Extension Theorem (which we will prove later.)

We can deduce an integral formula for derivatives

25 **Theorem 25**

Assume that f is analytic in an open disk Δ and that γ is a closed, piecewise smooth path in Δ. Let $k \geq 0$ be an integer. Then

$$n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta$$

for every $z \in \Delta \setminus |\gamma|$.

Proof: We have

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)} \, d\zeta$$

by the Cauchy Integral Formula.

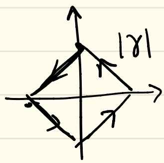
Since $n(\gamma,z)$ is locally constant repeated use of Proposition 19 we get

$$n(\gamma,z)\, f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta.$$

$\otimes$

<u>Ex</u> Calculate $\displaystyle\int_\gamma \frac{e^z + z^2\sin z}{z^3}\,dz$ where $|\gamma| = \{\, |x|+|y| = 1 \,\}$ positively oriented.

<u>Solution</u>:



$n(\gamma,0) = 1$

Let $f(z) = e^z + z^2\sin z$. We have

$$\int_\gamma \frac{e^z + z^2\sin z}{z^3}\,dz = \int_\gamma \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i\, f''(0)}{2!}.$$

$$f'(z) = e^z + 2z\sin z + z^2\cos z$$

$$f''(z) = e^z + 2\sin z + 2z\cos z + 2z\cos z - z^2\sin z$$

$$f''(0) = e^0 + \ldots = 1$$

$$\Longrightarrow \int_\gamma \frac{e^z + z^2\sin z}{z^3}\,dz = \pi i$$

Next we prove the Cauchy estimates which lets us relate the size of derivatives to the size of the function.

26 **Theorem 26** (Cauchy Estimates)

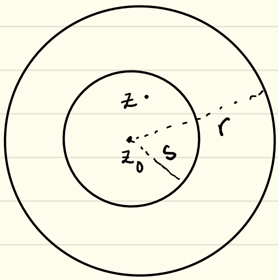
Assume that f is analytic in an open disk $\Delta = \Delta(z_0, r)$ and $|f(z)| \leq m$ when $z \in \Delta$ (m is some constant). Then, for each integer $k \geq 0$, we have

$$|f^{(k)}(z)| \leq \frac{k!\, m r}{(r - |z - z_0|)^{k+1}}$$

for $z \in \Delta$. In particular, $|f^{(k)}(z_0)| \leq \frac{k!\, m}{r^k}$.

Proof:

(Concentric circles)



Choose $z \in \Delta$ and s such that $0 \leq |z - z_0| < s < r$. Let $|\gamma_s|$ be $\{|\zeta - z_0| = s\}$ positively oriented

Then
$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

When $|\zeta - z_0| = s$ we see $|\zeta - z| = |\zeta - z_0 + z_0 - z| \geq$

$\geq \big| |\zeta - z_0| - |z_0 - z| \big| = s - |z - z_0|$

We get

$$|f^{(k)}(z)| \leq \left| \frac{k!}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right| \leq$$

$$\leq \frac{k!}{2\pi} \int_{\gamma_s} \frac{|f(\zeta)|}{(s - |z-z_0|)^{k+1}} |d\zeta| \leq \frac{k! m s}{(s - |z-z_0|)^{k+1}}$$

Let $s \to r$. We get

$$|f^{(k)}(z)| \leq \frac{k! m r}{(r - |z-z_0|)^{k+1}}$$

and if $z = z_0$ $\quad |f^{(k)}(z_0)| \leq \dfrac{k! m}{r^k}$ $\otimes$

From this we derive a surprising (maybe?) result.

27 **Theorem 27** (Liouville's Theorem)

Assume $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded (that is $|f(z)| \leq M$ for all $z \in \mathbb{C}$). Then f is constant.

Proof: Let $z \in \mathbb{C}$ and $\Delta(z, r) \subseteq \mathbb{C}$

Use Cauchy's Estimate with $k=1$

$$|f'(z)| \leq \frac{M}{r} \quad \text{and let } r \to \infty$$

Therefore $f'(z) = 0$ and f is constant. $\otimes$

Next we prove the Fundamental Theorem of Algebra

Theorem 28 (The Fundamental Theorem of Algebra)

28 Let $p(z)$ be a polynomial of degree $n \geq 1$. Then there is at least one $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Assume that $p(z) \neq 0$ for every $z \in \mathbb{C}$. Polynomials are entire functions and $f(z) = \frac{1}{p(z)}$ is entire if $p(z) \neq 0$ for every $z \in \mathbb{C}$

Write $p(z) = a_n z^n + \dots + a_1 z + a_0 \quad (a_n \neq 0)$ and note

$$|p(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \geq$$

$$\geq |z|^n \underbrace{\left(|a_n| - \frac{|a_{n-1}|}{|z|} - \dots \frac{|a_0|}{|z|^n} \right)}_{\to |a_n| \text{ as } |z| \to \infty}$$

Since $|a_n| > 0$ we see that $|p(z)| \to \infty$ as $|z| \to \infty$. Therefore $|f(z)| \to 0$ as $|z| \to \infty$.

This means $|f(z)| < 1$ when $|z| > R$ for some R. Since f is continuous on $\overline{\Delta}(0,R)$ it is bounded there and $|f(z)|$ is bounded on $\mathbb{C}$. By Liouville's Theorem f is constant and therefore also p is constant. This a contradiction. $\otimes$

As a consequence of this result and the Factor Theorem we get that every polynomial

$$p(z) = a_n z^n + \dots + a_1 z + a_0 \quad \text{can be written}$$

$$p(z) = a_n (z - z_1)(z - z_2) \dots (z - z_n)$$

where $p(z_j) = 0 \quad j = 1, \dots, n$ (the roots can be repeated)

The next result we will prove is the maximum principle (for analytic functions). For this we need the concept of a connected set. An open set X is said to be connected if it is not the union of 2 disjoint, nonempty open sets U and V. We formulate this as follows:

if U, V open, $U \cap V = \emptyset$, and $X = U \cup V$ then $U = \emptyset$ or $V = \emptyset$.

⌈For general sets X you should look for U and V open in the induced topology.⌋

Domain

In complex analysis an open connected set D is called a domain.

The Maximum Principle / Maximum Modulus Theorem

Let f be analytic in a domain D. Assume that there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for every $z \in D$. Then f is constant in D.

Proof: Assume that $|f(z_0)| = M$ and choose R such that $\Delta(z_0, R) \subseteq D$. Pick r such that $0 < r < R$.