As a consequence of this result and the Factor Theorem we get that every polynomial

$$p(z) = a_n z^n + \dots + a_1 z + a_0 \quad \text{can be written}$$

$$p(z) = a_n (z - z_1)(z - z_2) \dots (z - z_n)$$

where $p(z_j) = 0 \quad j = 1, \dots, n$ (the roots can be repeated)

The next result we will prove is the maximum principle (for analytic functions). For this we need the concept of a connected set. An open set X is said to be connected if it is not the union of 2 disjoint, nonempty open sets U and V. We formulate this as follows:

if U, V open, $U \cap V = \emptyset$, and $X = U \cup V$ then $U = \emptyset$ or $V = \emptyset$.

⌜For general sets X you should look for U and V open in the induced topology.⌟

Domain

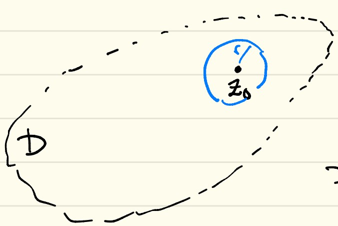
In complex analysis an open connected set D is called a domain.

The Maximum Principle / Maximum Modulus Theorem

Let f be analytic in a domain D. Assume that there exists a point $z_0 \in D$ such that $|f(z)| \le |f(z_0)|$ for every $z \in D$. Then f is constant in D.

Proof: Assume that $|f(z_0)| = M$ and choose R such that $\Delta(z_0, R) \subseteq D$.

Pick r such that $0 < r < R$.

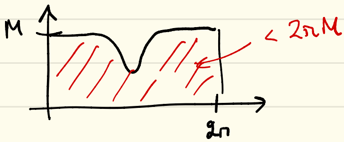


By the Cauchy estimates we have
$$|f(z_0)| \le \frac{1}{2\pi} \int_{\partial\Delta(z_0,r)} \frac{|f(z)|}{|z-z_0|} |dz| .$$

Parametrize $\partial\Delta(z_0,r)$ so that $z - z_0 = re^{i\theta}$ $\quad 0 \le \theta \le 2\pi$.

Then $$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0+re^{i\theta})|}{r} r\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\, d\theta$$

If $|f(z_0+re^{i\theta})| < M$ for some θ we get a contradiction



Hence $|f(z_0+re^{i\theta})| = M$ for all θ and $|f(z)| = M$ on $\Delta(z_0,r)$. This means $U = \{z \in D;\ |f(z)| = M\}$ is open and non-empty. Since $|f(z)|$ is continuous the set $V = \{z \in D;\ |f(z)| < M\}$ is open (general topology).

Now $D = U \cup V$ and since $U \neq \emptyset \Rightarrow V = \emptyset$.

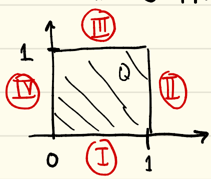
Hence $D = U = \{z \in D;\ |f(z)| = M\}$ $\otimes$

<u>Corollary</u>: If D is a bounded domain in the plane and $f : \overline{D} \to \mathbb{C}$ is continuous which is analytic in D. Then $|f|$ attains it's maximum on ∂D.

Proof: The maximum of $|f|$ in $\bar{D}$ exists since $\bar{D}$ is compact (closed & bounded) and $|f|$ is continuous. If $|f|$ attains the maximum inside D it is constant by the Maximum Principle and therefore also constant on $\bar{D}$. Then the maximum is attained on ∂D. $\otimes$

Ex Let $f(z) = z^2 - 2z$. Compute the maximal value of $|f(z)|$ in $Q = \{z;\ 0 \le x \le 1,\ 0 \le y \le 1\}$

Solution: We examine $|f(z)|$ on ∂Q.



On I: $0 \le x \le 1$, $y = 0$

$f(z) = f(x + i0) = x^2 - 2x$

$g(x) = |f(z)| = 2x - x^2$

$g'(x) = 2 - 2x = 0 \Leftrightarrow x = 1$ $\qquad g(0) = 0$ & $g(1) = 2 - 1 = 1$

On II: $x = 1$, $0 \le y \le 1$

$$f(z) = f(1 + iy) = (1+iy)^2 - 2(1+iy) = 1 + 2iy - y^2 - 2 - 2iy = -y^2 - 1$$

$h(y) = |f(z)| = 1 + y^2 \implies h(0) = 1$ & $h(1) = 2$

On III: $0 \le x \le 1$, $y = 1$

$$f(z) = f(x+i) = (x+i)^2 - 2(x+i) = x^2 + 2xi + i^2 - 2x - 2i$$
$$= x^2 - 2x - 1 + (2x-2)i$$

... maximum $= \sqrt{5}$

On IV: ... maximum $= \sqrt{5}$

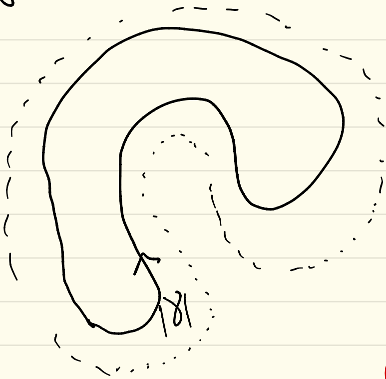
$$\implies \max_{z \in Q} |f(z)| = \sqrt{5}$$

The Global Cauchy Theorem

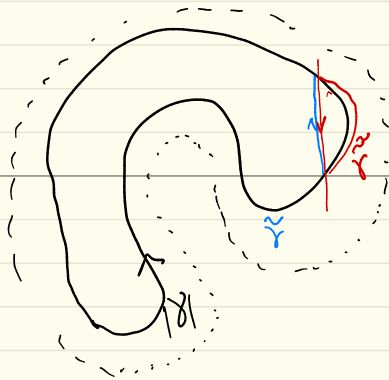
Remember that so far we have only shown that

$$\int_\gamma f(z)\,dz = 0 \qquad (f \text{ analytic})$$

when γ is a piecewise smooth closed path in a disk. It is desirable to remove this requirement. We will now explain how this is handled. This will not be a formal proof but hopefully the idea will be clear.



The idea is to modify the path step-by-step



$$\int_{\tilde{\tilde{\gamma}}} f(z)\,dz = 0$$

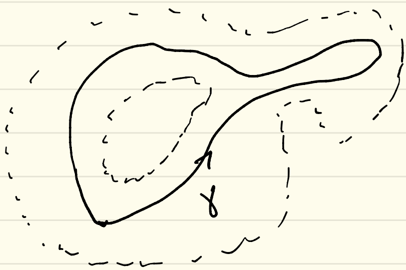
$$\int_{\tilde{\gamma}} f(z)\,dz = \int_\gamma f(z)\,dz$$

Keep changing the path until you can use the local version of Cauchy's Theorem. Hence

$$\int_\gamma f(z)\,dz = 0 \quad \text{(in this case)}$$

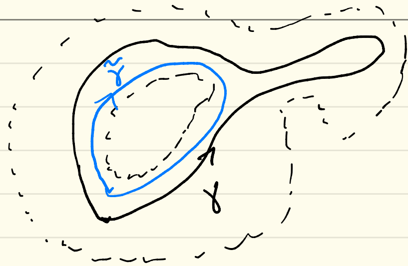
Is it always possible to change the path in this way?

No, for example, in this case it is not possible

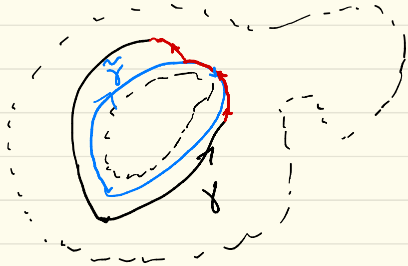


We can't shrink the path across the "hole"

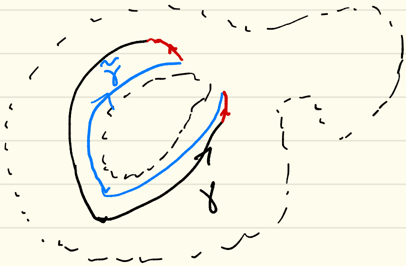
We also know that $\int_\gamma f(z)\,dz \neq 0$ if $f(z) = \frac{1}{z-a}$ when a is any point in the "hole"



However if we add a path going around the hole in the opposite direction we are "back in business"



The two paths can be merged into one if we cut apart on the red/blue portion.



Now we can continue to change the path and see that

$$\int_{\gamma \cup \tilde{\gamma}} f(z)\, dz = 0$$

Why did it work now? The reason is that $n(\gamma, a) \neq 0$ but $n(\gamma, a) + n(\tilde{\gamma}, a) = 0$ for any a in the hole.

In order to formulate the result in generality we need to introduce cycles of piecewise smooth closed paths (or cycles for short)

Given a collection $\gamma_1, \dots, \gamma_k$ of piecewise smooth closed paths in a set $A \subseteq \mathbb{C}$ a cycle in A is a k-tuple $\sigma = (\gamma_1, \dots, \gamma_k)$

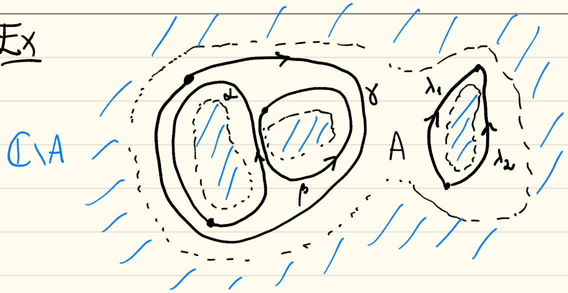
We define

$$\int_\sigma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \dots + \int_{\gamma_k} f(z)\,dz$$

and $n(\sigma, a) = n(\gamma_1, a) + \dots + n(\gamma_k, a)$

We say that two cycles σ_1 and σ_2 are homologous in A if $n(\sigma_1, a) = n(\sigma_2, a)$ for every $a \in \mathbb{C} \setminus A$. We say that a cycle σ is homologous to zero in A if $n(\sigma, a) = 0$ for every $a \in \mathbb{C} \setminus A$. Finally two non-closed piecewise smooth path γ_1 and γ_2 sharing initial and terminal points are homologous in A if $\sigma = \gamma_1 - \gamma_2$ is homologous to zero in A.

Ex



in A

The closed paths α, β, and γ are not homologous to zero. The cycles (α,β), (α,γ), and (β,γ) are not homologous to zero. However (α,β,γ) is homologous to zero in A. The paths λ_1 and λ_2 are not homologous in A.

We haven't proved the following but I hope the previous discussion is convincing.

Cauchy's Theorem (Global version)

Let σ be a cycle in an open set U. Then

$$\int_\sigma f(z)\,dz = 0 \quad \text{for every analytic function } f: U \to \mathbb{C}$$

if and only if σ is homologous to zero in U.

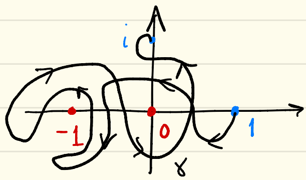
29 **Corollary 29**: If f is analytic in an open set $\mathcal{U}$ and if σ_0 and σ_1 are cycles that are homologous in $\mathcal{U}$, then
$$\int_{\sigma_0} f(z)\,dz = \int_{\sigma_1} f(z)\,dz .$$

30 **Corollary 30**: If f is analytic in an open set $\mathcal{U}$ and λ_0 and λ_1 are non-closed piecewise smooth paths in $\mathcal{U}$ that are homologous in $\mathcal{U}$, then
$$\int_{\lambda_0} f(z)\,dz = \int_{\lambda_1} f(z)\,dz .$$

Cauchy's Integral Formula (Global version)

Suppose that f is analytic in an open set $\mathcal{U}$ and that σ is a cycle that is homologous to zero in $\mathcal{U}$. Then
$$n(\sigma, z)\, f(z) = \frac{1}{2\pi i} \int_{\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$
for every $z \in \mathcal{U} \setminus |\sigma|$

<u>Ex</u> Evaluate $\int_\gamma \frac{z^2+z+1}{z^3+z^2}\,dz$ where γ is as in the figure



<u>Solution</u>: First we note that $f(z) = \frac{z^2+z+1}{z^3+z^2}$ is analytic in $\mathbb{C} \setminus \{0,-1\}$ (since $z^3+z^2 = z^2(z+1)$)

It is not really clear how to do this calculation. Lets experiment a little and see if we can make some progress. If we use partial fractions decomposition we see

$$\frac{z^2+z+1}{z^2(z+1)} = \frac{1}{z^2} + \frac{1}{z+1}.$$

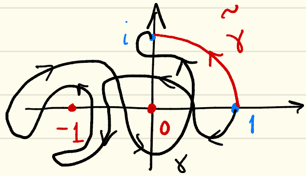
Now since $-\frac{1}{z}$ is a primitive function for $\frac{1}{z^2}$ we see

$$\int_\gamma f(z)\,dz = \left[-\frac{1}{z}\right]_1^i + \int_\gamma \frac{1}{z+1}\,dz = -\frac{1}{i} + 1 + \int_\gamma \frac{1}{z+1}\,dz$$

$$= i + 1 + \int_\gamma \frac{1}{z+1}\,dz$$

We now concentrate on $\int_\gamma \frac{1}{z+1}\,dz$.

$\frac{1}{z+1}$ is analytic in $\mathbb{C} \setminus \{-1\}$.



In $\mathbb{C}\setminus\{-1\}$ γ and $\tilde{\gamma}$ are homologous!
Also $\mathrm{Log}(z+1)$ is a primitive function for $1/(z+1)$ in $\mathbb{C}\setminus(-\infty,-1]$.

Therefore
$$\int_\gamma \frac{1}{z+1}\,dz = \int_{\tilde{\gamma}} \frac{1}{z+1}\,dz = \Big[\mathrm{Log}(z+1)\Big]_1^i =$$
$$= \mathrm{Log}(1+i) - \mathrm{Log}(2) = \ln\sqrt{2} + i\frac{\pi}{4} - \ln 2 =$$
$$-\frac{1}{2}\ln 2 + i\frac{\pi}{4}.$$

and
$$\int_\gamma \frac{z^2+z+1}{z^3+z^2}\,dz = 1+i-\frac{1}{2}\ln 2 + i\frac{\pi}{4} = 1-\ln\sqrt{2}+i\left(1+\frac{\pi}{4}\right).$$

Sequences and Series of Analytic Functions

Given a complex sequence $(z_n)_{n=1}^\infty$ we can form the partial sums $s_n = \sum_{k=1}^n z_k$. If the sequence $(s_n)_{n=1}^\infty$ converges to s (that is $\lim_{n\to\infty} s_n = s$) then we say that $\sum_{n=1}^\infty z_n$ is convergent with sum s. We write $s = \sum_{n=1}^\infty z_n = \lim_{n\to\infty} \sum_{k=1}^n z_k$.