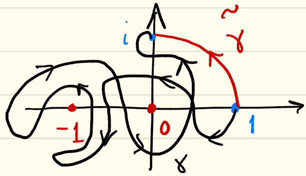


In $\mathbb{C}\setminus\{-1\}$ γ and $\tilde{\gamma}$ are homologous!
Also $\operatorname{Log}(z+1)$ is a primitive function for $1/(z+1)$ in $\mathbb{C}\setminus(-\infty,-1]$.

Therefore

$$\int_\gamma \frac{1}{z+1}\,dz = \int_{\tilde{\gamma}} \frac{1}{z+1}\,dz = \Big[\operatorname{Log}(z+1)\Big]_1^i =$$

$$= \operatorname{Log}(1+i) - \operatorname{Log}(2) = \ln\sqrt{2} + i\frac{\pi}{4} - \ln 2 = -\frac{1}{2}\ln 2 + i\frac{\pi}{4}.$$

and $\displaystyle\int_\gamma \frac{z^2+z+1}{z^3+z^2}\,dz = 1+i-\frac{1}{2}\ln 2 + i\frac{\pi}{4} = 1-\ln\sqrt{2}+i\left(1+\frac{\pi}{4}\right).$

Sequences and Series of Analytic Functions

Given a complex sequence $(z_n)_{n=1}^\infty$ we can form the partial sums $s_n = \sum_{k=1}^n z_k$. If the sequence $(s_n)_{n=1}^\infty$ converges to s (that is $\lim_{n\to\infty} s_n = s$) then we say that $\sum_{n=1}^\infty z_n$ is convergent with sum s. We write $s = \sum_{n=1}^\infty z_n = \lim_{n\to\infty} \sum_{k=1}^n z_k$.

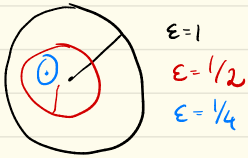
Cauchy sequence

Cauchy's Criterion for Convergence

Let $(s_n)_{n=1}^{\infty}$ be a complex sequence. We call $(s_n)_{n=1}^{\infty}$ a Cauchy sequence if for every $\varepsilon > 0$ there is N such that if $m \geq N$ and $k \geq N$ then $|s_m - s_k| < \varepsilon$.

31 **Theorem 31** A complex sequence $(s_n)_{n=1}^{\infty}$ is convergent iff it is a Cauchy sequence.

We will not prove this fundamental theorem. It is usually proven in Euclidean Spaces. However, lets try to see why it needs a proof.



How can we be sure there is an s that s_n approaches? This is the content of the theorem.

32 **Proposition 32** Let $\sum_{n=1}^{\infty} z_n$ be a complex series. Assume that

$$\sum_{n=1}^{\infty} |z_n| \text{ converges. Then}$$

$$\sum_{n=1}^{\infty} z_n \text{ is convergent.}$$

Proof: Since $\sum_{n=1}^{\infty} |z_n|$ is convergent the sequence of partial sums $\tilde{S}_n = \sum_{k=1}^{n} |z_k|$ is a Cauchy sequence (real sequences are also complex).

This implies that $s_n = \sum_{k=1}^{n} z_k$ forms a Cauchy sequence since

$$|s_n - s_m| = \left|\sum_{k=n}^{m} z_k\right| \leq \sum_{k=n}^{m} |z_k| =$$

$$= |\tilde{S}_m - \tilde{S}_n|.$$

Hence $\sum_{n=1}^{\infty} z_n$ is convergent. ⊗

We say that $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ is convergent. These are good to work with since we have many criteria for convergence for positive series from Differential and Integral Calculus 1.

Geometric series

<u>Ex</u> For which $z \in \mathbb{C}$ does $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$ converge?

<u>Solution</u>: We know (from one exercise) that

$$S_n = 1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad \text{if } z \neq 1$$

Since $\lim_{n\to\infty} z^{n+1} = 0$ if $|z| < 1$ and

$\lim_{n\to\infty} z^{n+1}$ divergent if $|z| \geq 1$ (and $z \neq 1$)

we see that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ if $|z| < 1$

and otherwise it is divergent.
($z = 1$ needs to be checked independently but $1 + 1 + 1^2 + 1^3 + \dots$ is clearly divergent)

Ex Does $\sum_{n=1}^{\infty} \frac{n(1+i)^n}{(2i)^n}$ converge or diverge?

Solution: We study $|z_n| = \left|\frac{n(1+i)^n}{(2i)^n}\right| = \frac{|n|\,|1+i|^n}{|2|^n} = \frac{n(\sqrt{2})^n}{2^n}$

Does $\sum_{n=1}^{\infty} |z_n|$ converge or diverge?

We can use for example the root criterion for positive series to try to answer this.

$\lim_{n\to\infty} \sqrt[n]{|z_n|} = \lim_{n\to\infty} \sqrt[n]{n}\,\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} < 1.$

Hence $\sum_{n=1}^{\infty} |z_n|$ converges and therefore $\sum_{n=1}^{\infty} \frac{n(1+i)^n}{(2i)^n}$ is absolutely convergent. Therefore $\sum_{n=1}^{\infty} \frac{n(1+i)^n}{(2i)^n}$ converges. $\otimes$

In general there is nothing called absolutely divergent. We can find series that converges but not absolutely ($\underline{Ex}$: $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$). Such a series is called conditionally convergent. They are quite peculiar since here the order of the terms matter for the value!

⌜This is however nothing compared to divergent series where you can find such "mythical beasts" such as

$1+2+3+4+5+6+\ldots = -\frac{1}{12}$!!! ⌟

For us it will be important to study complex series as functions of z. Questions about continuity and differentiability will be important. Lets begin with a toy problem to illustrate.

Ex Let $f_n : [0,1] \to \mathbb{R}$ be $f_n(x) = x^n$.

Then $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n =$

$$= \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$

All f_n are continuous but f is not!

We see that pointwise convergence is not "strong enough" to guarantee continuity of the limit function. What is needed is so called uniform convergence. The problem in the example above is that the convergence happens "at different speeds" at different points in $[0,1]$.

We need the concept of supremum (a generalisation of maximum) to formulate uniform convergence.

Let $A \subseteq \mathbb{R}$. The supremum is the smallest number $M \in \mathbb{R}$ such that $x \le M$ whenever $x \in A$.

If no such M exists then $\sup A = \infty$ if $A \neq \emptyset$ and $\sup A = -\infty$ if $A = \emptyset$

Ex: $A = [0,1]$ then $\sup A = 1$ ($\max A = 1$)

$A = (0,1)$ then $\sup A = 1$ ($\max A$ doesn't exist here)

$A = \mathbb{R}$ then $\sup A = \infty$

There is also a corresponding concept for "lower bounds" of A called infimum

Ex: $\inf [0,1] = 0$

$\inf (0,1) = 0$ etc.

Uniform convergence of a sequence of functions

Def: Let $A \subseteq \mathbb{C}$ and $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n : A \to \mathbb{C}$. Let $f : A \to \mathbb{C}$. We say that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on A if for every $\varepsilon > 0$ there is an N so that

$$\sup(|f_n(z) - f(z)| \,;\, z \in A) < \varepsilon$$

when $n \geq N$.

Notice first that uniform convergence implies that $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in A$.

33 Theorem 33

Assume that $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions on a set $A \subseteq \mathbb{C}$ that converges uniformly on A to f. Then f is continuous on A. Also if γ is a piecewise smooth path in A then

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz.$$

<u>Proof</u>: Take $z_0 \in A$. We need to show that $\lim_{z \to z_0} f(z) = f(z_0)$. Choose $\varepsilon > 0$. We need to find $\delta > 0$ so that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

$$(*)\quad |f(z) - f(z_0)| \le |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)|$$

$$\le |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|$$

Since (f_n) converges uniformly we can find N so that $|f(z) - f_n(z)| < \varepsilon/3$ for all $z \in A$ (also z_0!)

Since f_n is continuous we can find $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f_n(z) - f_n(z_0)| < \varepsilon/3$

Hence if $|z - z_0| < \delta$ by $(*)$ we get

$$|f(z) - f(z_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So f is continuous on A.

Next, we prove

$$\int_\gamma f(z)\,dz = \lim_{n \to \infty} \int f_n(z)\,dz.$$

(Notice: $\int_\gamma f(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz$) This tells us that you can switch order of lim and $\int$ if $f_n \to f$ uniformly

Now pick N so that $|f_n(z) - f(z)| < \dfrac{\varepsilon}{1 + l(\gamma)}$ for all $z \in A$ when $n \ge N$. (uniform convergence!)

Then

$$\left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right| \le \int_\gamma |f_n(z) - f(z)|\,|dz|$$

$$\le \frac{\varepsilon}{1 + l(\gamma)} \cdot l(\gamma) < \varepsilon \quad \text{when } n \ge N.$$

So $\displaystyle\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz$ $\otimes$

So we see that the limit function is continuous if it is the uniform limit of a sequence of continuous functions. What about analytic functions?

34 **Definition 34**

A set $K \subset \mathbb{C}$ is compact iff K is closed and bounded. (Bounded = subset of some $\Delta(0, r)$)

35 **Definition 35**

A sequence of analytic functions (f_n) on an open set U is said to converge normally to $f : U \to \mathbb{C}$ iff (f_n) converges uniformly to f on every compact $K \subset U$.

36 **Theorem 36**

Suppose that (f_n) is a sequence of analytic functions on an open set U that converges normally to f on U. Then f is analytic of U. Moreover the (derived) sequences $(f_n^{(k)})$ $(k = 1, 2, 3, \dots)$ converges normally in U to $f^{(k)}$.

Proof: First we prove that f is analytic. Take a rectangle $R \subseteq U$. Since f is continuous $\int_{\partial R} f(z)\,dz$ is well-defined. R is a compact set we get $\int_{\partial R} f(z)\,dz = \lim_{n\to\infty} \int_{\partial R} f_n(z)\,dz$ by Thm 33.

However since f_n is analytic $\int_{\partial R} f_n(z)\,dz = 0$

Therefore $\int_{\partial R} f(z)\,dz = 0$ and since this holds for every $R \subseteq U$ it follows that f is analytic by Morera's Theorem.

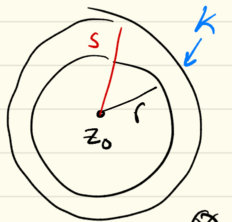
Lets show that $f_n^{(k)} \to f^{(k)}$ normally on $\mathcal{U}$.
It is enough to do for $k=1$ (the rest follows by induction). So we want show $f_n' \to f'$ uniformly on compacts $K \subseteq \mathcal{U}$. It is enough to show uniform convergence for disks $\overline{\Delta(z_0,r)} \subseteq \mathcal{U}$ since any K can be covered by finitely many $\overline{\Delta(z_0,r)}$
(This is the reason compact sets are important here)

Take $z_0 \in \mathcal{U}$ and $r>0$ such that $\overline{\Delta(z_0,r)} \subseteq \mathcal{U}$. We can find $s>r$ so that $\overline{\Delta(z_0,s)} \subseteq \mathcal{U}$ still. For $z \in \overline{\Delta(z_0,r)}$ we have

$$|f_n'(z)-f'(z)| = \left| \frac{1}{2\pi i} \int_{|\zeta-z_0|=s} \frac{f_n(\zeta)-f(\zeta)}{(\zeta-z)^2} d\zeta \right|$$

$$\leq \frac{1}{2\pi} \int_{|\zeta-z_0|=s} \frac{|f_n(\zeta)-f(\zeta)|}{|\zeta-z|^2} |d\zeta| \leq \frac{s}{(s-r)^2} \sup_K |f_n(\zeta)-f(\zeta)|$$

$\leq \varepsilon$ for all $z \in \overline{\Delta(z_0,r)}$
when n big enough



⊗

We have done much preparation but now we are almost ready for some real benefits.

Cauchy criterion for uniform convergence.

37

Series of functions

Given a sequences of functions $(f_n : A \to \mathbb{C})_{n=1}^{\infty}$ (defined on some set $A \subseteq \mathbb{C}$) we can form the sequence of partial sums $S_n(z) = \sum_{k=1}^{n} f_k(z)$.
All the concepts of convergence now translates to series via the partial sums. We have the criterion:

- $\sum_{n=1}^{\infty} f_n$ converges uniformly on A iff
- For every $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that $\sup\left(\left|\sum_{k=n}^{m} f_k(z)\right| ; m,n \geq N \text{ and } z \in A \right) < \varepsilon$

Theorem 37 (Weierstrass M-test)
Suppose that each term in a function series $\sum_{n=1}^{\infty} f_n$ is defined on a set A. Assume that there is M_n such that $|f_n(z)| \leq M_n$ for each $z \in A$ and also that $\sum_{n=1}^{\infty} M_n$ converges. Then

$\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly on A.

Proof: Fix $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ is convergent we can find $N(\varepsilon)$ so that $M_m + \dots + M_k < \varepsilon$ if $N \leq m \leq k$.
We get $|f_m(z) + \dots + f_k(z)| \leq |f_m(z)| + \dots + |f_k(z)| \leq$
$\leq M_m + \dots + M_k < \varepsilon$ for each $z \in A$.

Therefore first of all $\sum_{n=1}^{\infty} |f_n(z)|$ is convergent and we get absolute convergence for every $z \in A$.
Also by Cauchy's criterion we get uniform convergence in A.

One more thing about normal convergence: It is hard to check uniform convergence in any compact subset of an open set U. However it is enough to check uniform convergence in closed disks in U. This imply normal convergence (since any compact can be covered by a finite union of closed disks.)

Ex The series $\sum_{n=0}^{\infty} z^n$ defines an analytic function in $|z|<1$.

This is because on $|z|\leq r<1$ we see that $|z^n| = |z|^n \leq r^n$ and $\sum_{n=0}^{\infty} r^n$ converges for $0\leq r<1$.
Hence $\sum_{n=0}^{\infty} z^n$ converges normally on $|z|<1$ and since the partial sums are analytic the limit series is as well. In fact, we already know that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{in } |z|<1.$$

It is interesting to note that Theorem 36 implies that we can differentiate termwise and get valid series of functions! That is,

$$\sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2} \quad \text{in } |z|<1 \quad \text{and so on.}$$

Ex Show that $\sum_{n=1}^{\infty} n^{-z}$ defines an analytic function in the half-plane $\{z\in\mathbb{C};\ \mathrm{Re}(z)>1\}$.

We show that $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly when $\mathrm{Re}(z) \geq \sigma > 1$. Study $|n^{-z}| = |(e^{\ln n})^{-z}| = n^{-\mathrm{Re}(z)} \leq n^{-\sigma}$

We know that $\sum_{n=1}^{\infty} n^{-\sigma}$ converges (it is a p-series with $p>1$) So Weierstrass M-test gives the result. Normal convergence follows and $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is analytic. This is the Riemann zeta-function (representation valid in $\mathrm{Re}(z)>1$)
We also get

$$\zeta'(z) = -\sum_{n=1}^{\infty} (\ln n)\, n^{-z} \quad \text{(with normal convergence)}$$

Taylor Series

A Taylor series is a function series of the following type. Take a sequence of complex numbers $(a_n)_{n=0}^{\infty}$ and $z_0 \in \mathbb{C}$. Then

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{is a Taylor series}$$

(or power series) centered at z_0 with coefficients (a_n). It is interesting to determine for which z this converges (and also to see what type of convergence we have). The following will be important.

__Def__ The "number" $\rho \in [0,\infty) \cup \{\infty\}$ defined as

$$\rho = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \right)^{-1} \quad \text{is called}$$

the radius of convergence of the Taylor series.

__Convention:__ $\frac{1}{0} = \infty$ & $\frac{1}{\infty} = 0$