Exercise session 1

1. a) Assume that $z \neq 1$. Prove that

$$\sum_{j=0}^{n} z^j = \frac{1-z^{n+1}}{1-z}$$

b) Find all solutions $z \in \mathbb{C}$ such that

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0$$

where $n > 1$ is an integer.

Solution: a) Put $s = \sum_{j=0}^{n} z^j = 1 + z + \dots + z^n$

Multiply this equality by z.

We get $zs = z + z^2 + \dots + z^n + z^{n+1}$

Therefore $(1-z)s = 1 - z^{n+1}$

and

$$s = \frac{1-z^{n+1}}{1-z} \quad (\text{since } 1-z \neq 0)$$

b) First notice that $z=1$ does not solve the equation.

$$1 + z + z^2 + \dots + z^{n-1} = \frac{1-z^n}{1-z} = 0$$

So $\quad 1+z+z^2+\dots+z^{n-1}=0 \quad$ iff $\quad z^n=1$
(and $z\neq 1$)

Introduce polar coordinates

$$z = re^{i\theta} \qquad z^n = r^n e^{in\theta} = 1\cdot e^{i2\pi k}$$

$k\in\mathbb{Z}$

$$\Rightarrow r=1 \;\&\; n\theta = 2\pi k \qquad k\in\mathbb{Z}$$

$$\theta = \frac{2\pi k}{n} \qquad k=1,2,\dots,n-1$$

We get that

$$z = e^{i\frac{2\pi k}{n}} \qquad k=1,2,\dots,n-1$$

all solve

$$1+z+\dots+z^{n-1}=0$$

These are the only solutions. $\varnothing$

(2) For $z\in\mathbb{C}$ we have either

$\sqrt{z^2}=z$ or $\sqrt{z^2}=-z$ (Recall $\sqrt{\cdot}$ is the principal square root).

[$\sqrt{z^2}=w$ solves $z^2=w^2$]

For which z does $\sqrt{z^2}=z$ hold?

For which z does $\sqrt{z^2}=-z$ hold?

Solution: Recall that $\sqrt{w}=e^{\frac{1}{2}\mathrm{Log}(w)}$

where Log is the principal logarithm

and

$$\mathrm{Log}(w) = \ln|w| + i\,\mathrm{Arg}(w)$$

where $\mathrm{Arg}(w)\in(-\pi,\pi]$

So $\sqrt{z^2} = e^{\frac{1}{2}\mathrm{Log}(z^2)} = e^{\frac{1}{2}\ln|z^2| + i\frac{1}{2}\mathrm{Arg}(z^2)}$

$$= |z| e^{i\frac{\mathrm{Arg}(z^2)}{2}}$$

Also

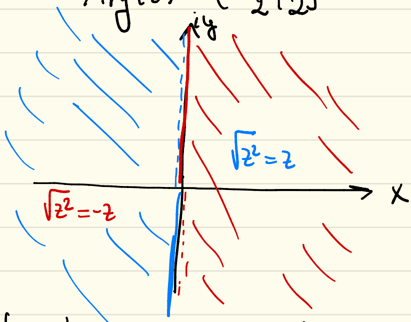
$$z = |z| e^{i\,\mathrm{Arg}(z)}$$

So $\sqrt{z^2} = z$ when $\mathrm{Arg}(z^2) = 2\,\mathrm{Arg}(z)$

This is true if $2\,\mathrm{Arg}(z) \in (-\pi, \pi]$

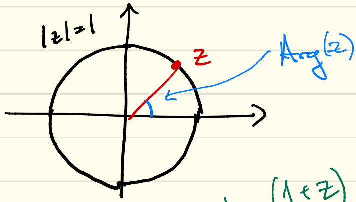
or if

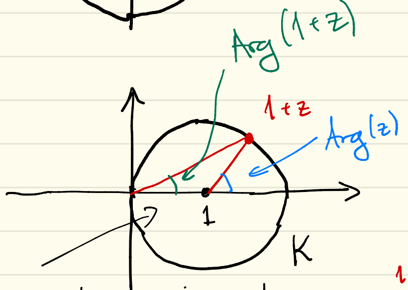
$$\mathrm{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



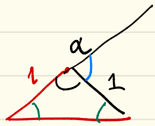
③ Verify that $2\,\mathrm{Arg}(1+z) = \mathrm{Arg}\, z$ when $|z|=1$, but $z \neq -1$. (Hint: What is the set $K = \{1+z;\ |z|=1\}$?)

Solution:





This triangle is isosceles



Therefore $2\,\mathrm{Arg}(1+z)+\alpha=\pi$. Also

$$\mathrm{Arg}(z)+\alpha=\pi$$

and we get $2\,\mathrm{Arg}(1+z)=\mathrm{Arg}(z)$ if $|z|=1$, but $z\neq -1$. $\mathrm{Arg}(1+(-1))$ undefined

4 If n is a positive integer, prove that

$$1+\cos\theta+\cos 2\theta+\dots+\cos(n\theta)=\frac{1}{2}+\frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}$$

unless θ is a multiple of 2π.

Solution: We know

$$e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$$

and

$$e^{-ik\theta} = \cos(k\theta) - i\sin(k\theta)$$

Therefore $\cos(k\theta) = \dfrac{e^{ik\theta} + e^{-ik\theta}}{2}$

and $\sin(k\theta) = \dfrac{e^{ik\theta} - e^{-ik\theta}}{2i}$

We get $1 + \cos\theta + \dots + \cos(n\theta) = \frac{1}{2}\left(\sum_{k=0}^{n} (e^{i\theta})^k + \sum_{k=0}^{n} (e^{-i\theta})^k\right)$

$$= \frac{1}{2}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-i(n+1)\theta}}{1 - e^{-i\theta}}\right) =$$

$$= \frac{1}{2}\left(\frac{\cancel{e^{i\frac{1}{2}\theta}}\left(e^{-i\frac{1}{2}\theta} - e^{i\left(\frac{2n+1}{2}\right)\theta}\right)}{\cancel{e^{i\frac{1}{2}\theta}}\left(e^{-i\frac{1}{2}\theta} - e^{i\frac{1}{2}\theta}\right)} + \frac{\cancel{e^{-i\frac{1}{2}\theta}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{2n+1}{2}\theta}\right)}{\cancel{e^{-i\frac{1}{2}\theta}}\left(e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}\right)}\right)$$

$$= \frac{1}{2}\left(\frac{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} + \frac{e^{i\left(\frac{2n+1}{2}\right)\theta} - e^{-i\left(\frac{2n+1}{2}\right)\theta}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}\right) =$$

$$= \frac{1}{2}\left(1 + \frac{\sin\left(\frac{2n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}\right) = \frac{1}{2} + \frac{\sin\left((2n+1)\theta/2\right)}{2\sin(\theta/2)}$$