Exercise Session 2

1. Express the following in the form $x+iy$:

a) $\mathrm{Log}(-e^2)$ b) $(-1)^i$

c) $i^{\sqrt{2i}}$

<u>Solution</u>: a) $\mathrm{Log}(-e^2) = \ln|-e^2| + i\,\mathrm{Arg}(-e^2)$

$$= \ln(e^2) + i\pi = 2 + i\pi.$$

b) $(-1)^i = e^{i\,\mathrm{Log}(-1)} = e^{i(\ln|-1| + i\,\mathrm{Arg}(-1))} =$

$$= e^{i(0+i\pi)} = e^{-\pi}$$

c) Lets start with $\sqrt{2i} = \left(e^{\mathrm{Log}(2i)}\right)^{1/2} =$

$$= e^{\frac{1}{2}(\ln|2i| + i\,\mathrm{Arg}(2i))} = e^{\frac{1}{2}(\ln 2 + i\frac{\pi}{2})} =$$

$$= \sqrt{2}\cdot e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i$$

So $i^{\sqrt{2i}} = e^{\sqrt{2i}\cdot \mathrm{Log}(i)} = e^{\sqrt{2i}\left(i\frac{\pi}{2}\right)} =$

$$= e^{(1+i)i\frac{\pi}{2}} = e^{-\frac{\pi}{2}}\cdot e^{i\frac{\pi}{2}} =$$

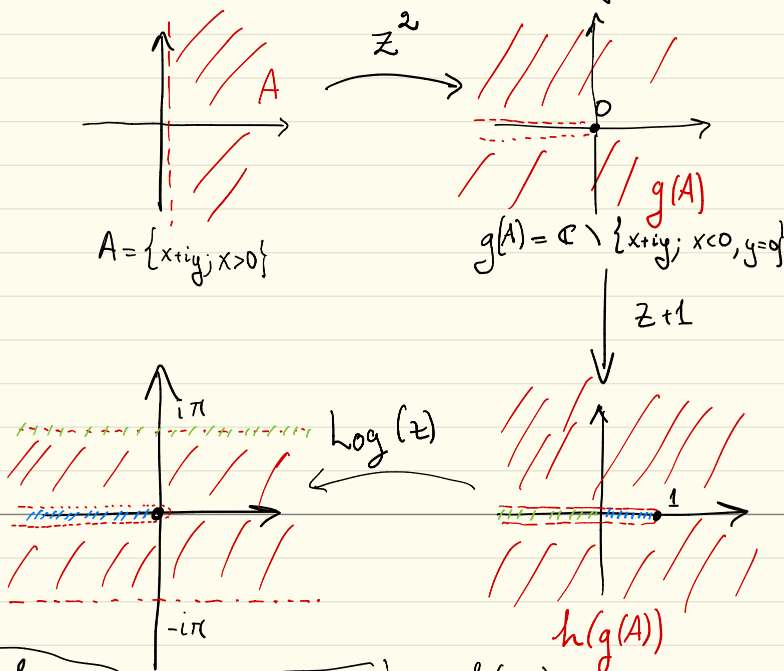
$$= e^{-\frac{\pi}{2}}\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i e^{-\frac{\pi}{2}}.$$

(2) Let $A = \{z;\ \mathrm{Re}(z) > 0\}$ and let $f: A \to \mathbb{C}$ be given by $f(z) = \mathrm{Log}(1+z^2)$

Determine the range $B = f(A)$.

(Hint: Look at f as a composition of $g(z) = z^2$, $h(z) = z+1$, and $k(z) = \mathrm{Log}\, z$)

<u>Solution</u>: Note that $f = k \circ h \circ g$. Then



$B = f(A)$

$h(g(A)) = \mathbb{C} \setminus \{x+iy;\ x<1,\ y=0\}$

The colored parts shows the corresponding parts "being missed" green, blue

We see that $B = f(A) =$

$$= \{x+iy;\ 0<y<\pi\} \cup \{x+iy;\ -\pi<y<0\} \cup \{x+iy;\ y=0,\ x>0\}$$

(3) Show that the locus of points in the complex plane satisfying

$$A z\bar{z} + Bz + \bar{B}\bar{z} + C = 0$$ in which

A and C are real numbers satisfying

$|B|^2 - AC > 0$ is a circle when $A \neq 0$

and a line if $A = 0$. (Converse also holds but you don't need to show this.)

Solution: We can assume $A \geq 0$ (multiply by -1 if needed)

If $A > 0$ we have

$$0 = A z\bar{z} + Bz + \bar{B}\bar{z} + C = A\left(z\bar{z} + \frac{B}{A}z + \frac{\bar{B}}{A}\bar{z} + \frac{C}{A}\right)$$

$$= A\left(\left(z + \frac{\bar{B}}{A}\right)\left(\bar{z} + \frac{B}{A}\right) - \frac{B\bar{B}}{A^2} + \frac{C}{A}\right) =$$

$$\underset{A=\bar{A}}{=} A\left(\left|z + \frac{\bar{B}}{A}\right|^2 - \frac{|B|^2 - CA}{A^2}\right)$$

Hence $\left|z - \left(-\frac{\bar{B}}{A}\right)\right|^2 = \frac{|B|^2 - CA}{A^2}$.

Since $|B|^2 - CA > 0$ this is a circle with center $-\frac{\bar{B}}{A}$ and radius $\frac{\sqrt{|B|^2 - CA}}{A}$ (we assumed $A > 0$ here)

So in general if $A \neq 0$ then

$$A z\bar{z} + Bz + \bar{B}\bar{z} + C = 0$$ is a circle with center $-\frac{\bar{B}}{|A|}$ and radius $\frac{\sqrt{|B|^2 - AC}}{|A|}$

Finally assume $A = 0$. Then we get

$$Bz + \bar{B}\bar{z} + C = 0$$

$$Bz + \bar{B}\bar{z} = -C$$

$$2\,\mathrm{Re}(Bz) = -C$$

Assume $B = \alpha + i\beta$

$$Bz = (\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\alpha y + \beta x)$$

So we get $\alpha x - \beta y = -\frac{C}{2}$ which is a line in the plane.

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④ Compute: a) all square roots of $-1 + i\sqrt{3}$

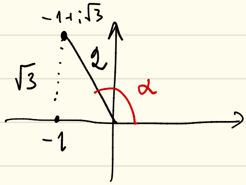
b) all cube roots of -8

Solution: a) We are interested in all complex numbers z satisfying $z^2 = -1 + i\sqrt{3}$.

Use polar coordinates

$$z = re^{i\theta} \Rightarrow z^2 = r^2 e^{i2\theta}$$

Also



$$\alpha = \pi - \arccos\left(\frac{\sqrt{3}}{2}\right) =$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

So $r^2 e^{i2\theta} = 2e^{i\left(\frac{2\pi}{3}+2\pi n\right)}$ $\quad n\in\mathbb{Z}$

Then $r=\sqrt{2}$ and $\theta = \frac{\pi}{3}+\pi n$ $\quad n\in\mathbb{Z}$

We get $z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$= \frac{\sqrt{2}}{2}+i\frac{\sqrt{6}}{2}$$

and $z_2 = \sqrt{2}\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right) = -\sqrt{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) =$

$$= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}$$

b) We solve

$$z^3 = -8$$

in the same way as a

$$z = re^{i\theta} \Rightarrow z^3 = r^3 e^{i3\theta}$$

and $-8 = 8e^{i(\pi+2\pi n)}$.

Therefore

$$r^3 = 8 \quad \text{and} \quad \theta = \frac{\pi}{3}+\frac{2\pi n}{3} \quad n\in\mathbb{Z}$$

We get $z_1 = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 1+i\sqrt{3}$

, $z_2 = 2(\cos\pi + i\sin\pi) = -2$, and

$z_3 = 2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = 1-i\sqrt{3}$