of 10 cm, the lower limiting frequency $f_c = \omega_c/2\pi$ of the horn is 100 Hz, and the medium is air. Obviously, the horn significantly improves the impedance match of the sound source to the medium. The radiation resistance of the piston fitted with the horn grows relatively fast when the frequency is above the cut-off frequency, and it exceeds that of the free piston in a wide frequency range. This advantage is paid for by zero radiation when driven below the cut-off frequency.

Again, it should be noted that these properties are strictly valid for the infinitely long horn only. Even with this limitation the advantages of horns are so prominent that they find wide technical application.

8.5 Higher order wave types

Up to this point it has been supposed that the lateral dimensions of pipes are significantly smaller than the acoustic wavelength – an assumption which may be rather questionable for the horn as has been described in the preceding section.

Now we omit this presupposition and investigate whether there are other waveforms – apart from the plane wave – which can be propagated in a pipe or – more generally – in an acoustical waveguide. We restrict this discussion to pipes of constant cross section.

At first, let us consider two plane harmonic waves of equal frequency propagating in directions which are inclined by the arbitrary angle $\pm\varphi$ to the x-axis. Their wave normals are represented in Figure 8.12 as arrows.

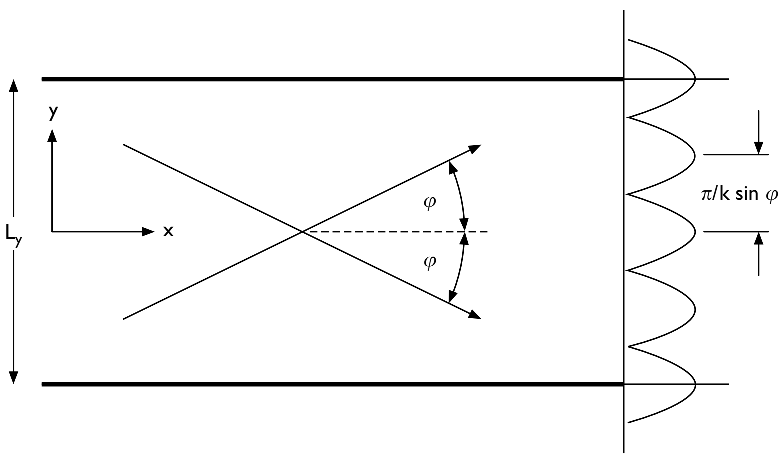

Figure 8.12 Two crossed plane waves. Right side: transverse distribution of the resulting sound pressure amplitude.

According to eq. (4.11) the sound pressures associated with them are, with $\alpha = \varphi$, $\cos\beta = \sin\varphi$, $\cos\gamma = 0$:

$$p_{1,2}(x, y) = \hat{p}e^{jk(-x\cos\varphi \pm y\sin\varphi)} \tag{8.44}$$

By adding p_1 and p_2 and using Euler's formula we obtain for the total sound pressure:

$$p(x, y) = 2\hat{p}\cos(ky\sin\varphi) \cdot e^{-jkx\cos\varphi} \tag{8.45}$$

Here as well as in the following expressions the exponential factor $\exp(j\omega t)$ containing the time dependence has been omitted. Equation (8.45) represents a wave progressing in the positive x-direction with angular wavenumber $k' = k\cos\varphi$. Obviously, this is not a plane wave since its amplitude is modulated with respect to the y-coordinate in the same way in a standing wave. Maximum sound pressure amplitudes occur whenever $ky \cdot \sin\varphi$ is an integral multiple of π, that is, in the planes $y = m\pi/(k\sin\varphi)$ where m is an integer. At these positions the vertical component of the particle velocity is zero. Accordingly, the sound field would not be disturbed by replacing two of this planes with rigid surfaces. Conversely, if the distance L_y of two parallel and rigid surfaces is given, a sound wave given by eq. (8.45) can propagate between them if $kL_y \cdot \sin\varphi = m\pi$; this relation defines the angle φ in eq. (8.45). From this condition we obtain the angular wavenumber relevant for the propagation in the x- direction:

$$k' = k\cos\varphi = \frac{\omega}{c}\sqrt{1 - \left(\frac{m\pi c}{\omega L_y}\right)^2} \tag{8.46}$$

It is real if the angular frequency ω exceeds a certain cut-off frequency ω_m which depends on the order m and distance L_y of both surfaces, that is, on the height of the 'channel':

$$\omega_m = \frac{m\pi c}{L_y} \tag{8.47}$$

The wave represented by eq. (8.45) is called a mth order wave type or wave mode. The wave characterised by $m = 0$ is the fundamental wave, and it is identical with the plane wave of our earlier sections and has a cut-off frequency zero. The wave velocity $c' = \omega/k'$ of the wave type of order m is

$$c' = \frac{c}{\sqrt{1 - (\omega_m/\omega)^2}} \tag{8.48}$$

This formula agrees with eq. (8.40) for the exponential horn and shows the same frequency dependence. Figure 8.13a plots the wave speed (which

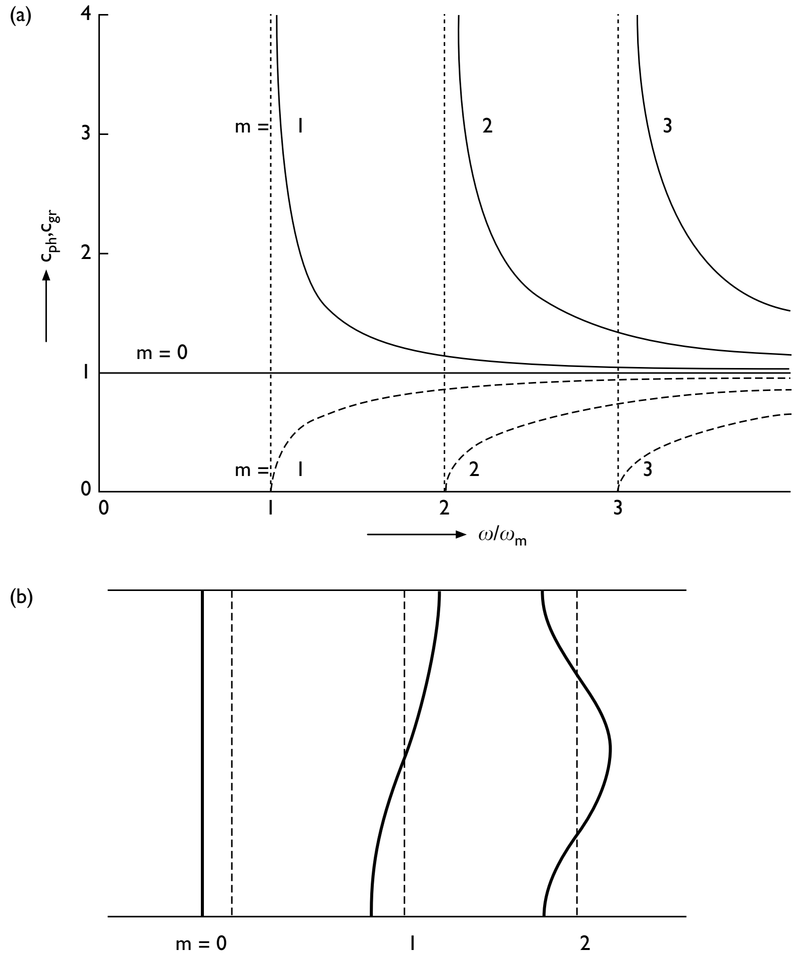

Figure 8.13 Higher order wave types in a two-dimensional waveguide: (a) speed of propagation (solid lines: phase velocity, broken lines: group velocity) as a function of the angular frequency, (b) transverse distribution of the sound pressure amplitude.

should be more correctly referred to as the 'phase velocity' as will be explained in the following section) as a function of the frequency (solid lines). For $m > 0$ it is always larger than the free field sound velocity c which it approaches asymptotically at very high frequencies.

Now we can represent the sound pressure in the mth mode by

$$p(x, y) = 2\hat{p}\cos\left(\frac{m\pi y}{L_y}\right) \cdot e^{-jk'x} \tag{8.49}$$

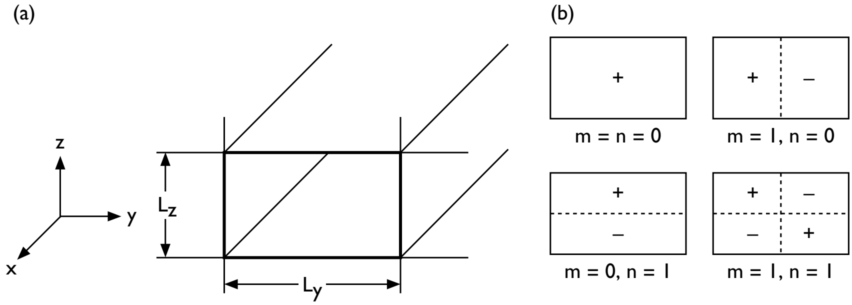

Figure 8.14 Sound propagation in a duct with rectangular cross section: (a) sketch of the duct, (b) nodal planes (section) of a few wave types (broken lines), m, n: integers in eq. (8.51a).

The lateral distribution of the sound pressure amplitude is shown in Figure 8.13b.

The preceding considerations can be extended without difficulties to a one-dimensional waveguide, that is, to a rigid-walled rectangular duct. For this purpose we imagine a second pair of rigid planes with a mutual distance L_z arranged perpendicular to the z-axis (see Fig. 8.14a).

As before we compose the sound field of two waves after eq. (8.49), however not travelling into the x-direction. Instead, they propagate in directions which are parallel to the x-z-plane but subtend the angles $\pm\varphi'$ with the x-axis. Hence, analogous to eq. (8.44):

$$p_{1,2}(x, y, z) = 2\hat{p}\cos\left(\frac{m\pi y}{L_y}\right) \cdot e^{-jk'(x\cos\varphi' \pm z\sin\varphi')} \tag{8.50}$$

The same considerations as in the derivation of eq. (8.49) lead to the expression

$$p(x, y, z) = 4\hat{p}\cos\left(\frac{m\pi y}{L_y}\right)\cos\left(\frac{n\pi z}{L_z}\right) \cdot e^{-jk''x\cos\varphi'} \tag{8.51a}$$

for the sound pressure, with n denoting a second integer. Now the wave field consists of standing waves with respect to the y- and the z-direction and it travels along the x-direction with the angular wave number:

$$k'' = \frac{\omega}{c}\sqrt{1 - (\omega_{mn}/\omega)^2} \tag{8.52}$$

Here we introduced the cut-off frequency:

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{L_y}\right)^2 + \left(\frac{n\pi c}{L_z}\right)^2} \tag{8.53}$$

A particular wave mode is characterised by two integers m and n; it is a progressive wave only if the driving frequency ω is above this cut-off frequency. For $\omega < \omega_{nm}$ the wavenumber k'' becomes imaginary; according to eq. (8.51a) this corresponds to a pressure oscillation with constant phase and with an amplitude decaying (or growing) exponentially with x. Below the lowest non-zero cut-off frequency the fundamental wave with $m = n = 0$ is the only propagating wave mode. Depending on whether L_y or L_z is the larger lateral dimension this cut-off frequency is either ω_{10} or ω_{01}. Or expressed in terms of the free field wavelength λ: the range where only the fundamental wave can be propagated is given by:

$$\lambda > 2 \cdot \mathrm{Max}\{L_y, L_z\} \tag{8.54}$$

Figure 8.14b presents an overview of the pressure distributions within the channel associated with some modes. The dotted lines indicate nodal planes where the pressure amplitude is zero at any time. They separate regions with opposite phase as marked by the signs. Both indices m and n indicate the number of nodal planes in the respective direction. Again, the wave velocity is given by eq. (8.48) after replacing ω_m with ω_{nm}. The same holds for Figure 8.13a, however, the limiting frequencies are no longer equidistant in this case.

The formulae derived earlier are valid in a slightly modified form for channels with soft boundaries the wall impedance of which is zero. In eqs. (8.49) and (8.51) we have just to replace the cosine functions with sine functions:

$$p(x, y, z) = 4\hat{p} \sin\left(\frac{m\pi y}{L_y}\right) \sin\left(\frac{n\pi z}{L_z}\right) \cdot e^{-jk'' x \cos\varphi'} \tag{8.51b}$$

because this time it is the sound pressure which has to vanish along the walls of the channel and not the normal components of the particle velocity. However, there exists no fundamental wave in a channel with soft walls since; according to eq. (8.51b) the sound pressure vanishes everywhere if m or n is zero. Waveguides with soft boundaries cannot be realised for gaseous media, but for liquid ones. Thus the surface of water, viewed from inside the water, has nearly zero impedance; furthermore, almost soft boundaries can be made with porous plastics with the pores filled with air.

Now we return to sound propagation in gas-filled waveguides. Most important are pipes with circular cross section for the transport of air or other gases. To calculate the sound propagation in such pipes one has to apply an

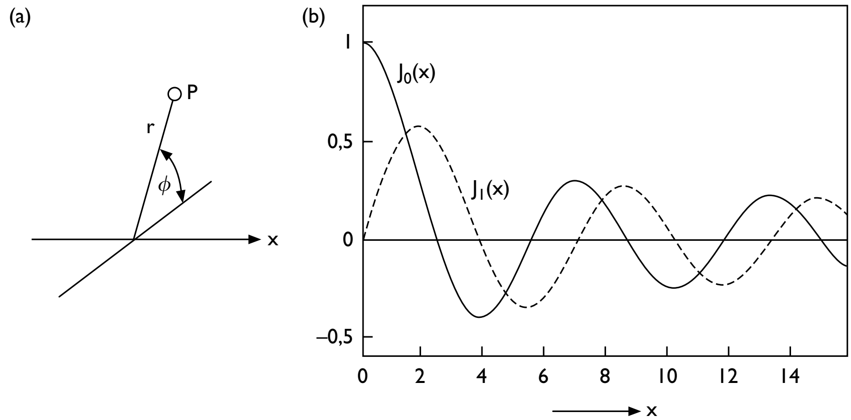

Figure 8.15 (a) Cylindrical coordinates, (b) the lowest order Bessel functions J_0 and J_1.

approach which we could have chosen as well for rectangular waveguides: it starts with the wave equation (3.21) expressed in coordinates which are appropriate to the geometry of the tube. In the present case these are cylindrical coordinates (see Fig. 8.15a). One coordinate axis coincides with the pipe axis, and we denote it as earlier with x. Furthermore, the position of a point P is characterised by its perpendicular distance r from the axis and by the angle between r and a fixed reference line which is also perpendicular to the axis. After specifying the wave equation in this way, the next step is to adapt its general solution to the boundary condition prescribed at the wall. As a result one arrives at the following expression for the sound pressure in a rigidly walled pipe:

$$p(x, r, \phi, t) = AJ_m\left(\nu_{mn}\frac{r}{a}\right) \cdot \cos(m\phi) \cdot e^{j(\omega t - k''x)} \tag{8.55}$$

Here J_m is the Bessel function of mth order which we encountered already in Subsection 5.8.2 (see eq. (5.39)). The first two of these functions, namely, J_0 and J_1, are plotted in Figure 8.15b. Each of them – and the same holds for higher order Bessel functions – has infinitely many maxima and minima. One of these must coincide with the pipe wall since in these points the derivative of the Bessel function and hence the radial component of the particle velocity vanishes. To achieve this, r/a in the argument of the Bessel function is multiplied with a number ν_{mn} which is the nth zero (starting with $n = 0$) of the derivative of the mth order Bessel function. Table 8.1 lists some of these zeros. Again, the angular wavenumber with respect to propagation along the

Table 8.1 Characteristic values ν_{mn} in eq. (8.55)

Order m of Bessel function	$n = 0$	$n = 1$	$n = 2$
0	0	3.832	7.015
1	1.841	5.331	8.526
2	3.054	6.706	9.970

axis is given by eq. (8.52); the cut-off frequency of the corresponding wave mode is

$$\omega_{mn} = \nu_{mn}\frac{c}{a} \tag{8.56}$$

Figure 8.16 shows the nodal surfaces for the lowest wave modes after eq. (8.55); the presentations are ordered after increasing cut-off frequencies. The nodal surfaces associated with the number m are concentric cylinders, and the other ones are planes containing the axis of the tube. The lowest cut-off frequency is that with $m = 1$ and $n = 0$, hence the frequency range in which the fundamental wave is the only one which can be propagated is characterised by

$$\lambda \geq 3.41 \cdot a \tag{8.57}$$

The earlier discussion provided some insight into the structure and properties of possible wave modes in a pipe. Whether these modes are actually excited by a certain sound source and participate in the transport of sound energy is a different question which cannot be answered without knowing the kind and position of the sound source. If the source consists of a rigid oscillating piston forming the termination of the pipe, we expect that the fundamental wave will be generated almost exclusively. However, if it is a point source located on the axis of a cylindrical pipe, only modes with $m = 0$ will be produced, since these are the only ones without any nodal surfaces containing the axis. If the point source is in an asymmetric position we have to reckon with the excitation of all wave types the cut-off frequencies of which are below the driving frequency, and their relative strengths depend on the position of the source.

8.6 Dispersion

In this chapter we have encountered two cases in which the wave velocity depends on the sound frequency, namely, the exponential horn and the higher order wave types in pipes or ducts. This phenomenon which is not restricted to acoustical waves is known as ‘dispersion’. We shall have a somewhat closer look at it in this section.