8

Horns and Stepped Ducts

The wave equations derived in the preceding chapter allow for calculation of arbitrary sound fields with any possible, physically meaningful boundary conditions. We had restricted ourselves to one-dimensional waves so far. These can, for instance, be observed in tubes with a diameter being small compared to the wavelength, that is $d \ll \lambda$. This condition guarantees that no other waveforms than axial ones can propagate in the tube. One-dimensional propagation also means that all wave planes perpendicular to the axial direction are planes of constant phase.

In the following, ducts shall be considered where the diameter varies with x. In other words, the area function $A = f(x)$ is no longer constant. Nevertheless, the condition $d \ll \lambda$ shall still holds. Two cases – depicted in Fig. 8.1 – will be discussed.

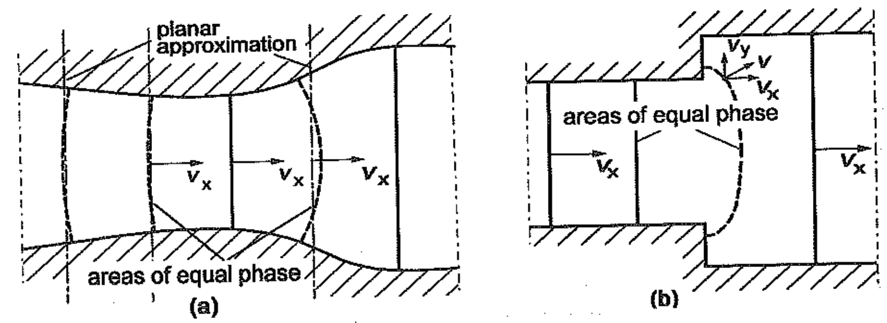

Fig. 8.1. Two types of ducts with non-constant area function. **(a)** continuous variation of the cross-sectional area, **(b)** stepped variations

- *Continuous variation* of the cross area. As long as this variation is only gradual compared to the wavelength, it is still justified to assume one-dimensional, axial propagation. Radial propagation can then be neglected

- *Step-like mutations* of the cross area as a function of x – so-called *stepped ducts*. Very close to the position where the step occurs, we certainly have radial components of the particle velocity. Yet, as radial waves cannot propagate, they can be neglected already at small distances away from the step. In fact, there we have plane waves again. If we look at the cross area just Δx in front of the step and again Δx behind it, we can state that the axial component of the volume velocity, $\underline{q} = A\,\underline{v}$, is the same in both cross sections. Thus, in our calculations, we can neglect any *modal dispersion* in the immediate vicinity of the step position and set the volume velocity at both sides of the step to equal

8.1 *Webster*'s Differential Equation – the Horn Equation

This chapter deals with the condition where the area function varies only gradually, and perpendicular areas are areas of approximately constant phase. This case is captured by the so-called *Webster equation* or *Horn equation*. Figure 8.2 illustrates the derivation of this differential equation. Please consider the elementary volume between the cross areas at x and $x{+}\mathrm{d}x$.

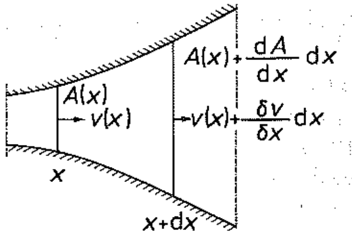

Fig. 8.2. Cross-section of a *horn*, i.e. a duct with gradually increasing cross area

The state equation and *Euler*'s equation are applied in their original form, namely,

$$\partial\varrho = \kappa_{=}\,\varrho_{=}\,\partial p \quad \text{and} \quad -\frac{\partial p}{\partial x} = \varrho_{=}\frac{\partial v}{\partial t}\,, \tag{8.1}$$

whereby, to be sure, $p = p(t,x)$ and $v = v(t,x)$ are functions of both time and space. In the continuity equation, the non-constant area function, A(x), must be reconsidered in the following way. The inflowing mass is

$$\mathrm{d}m_{\text{in}} = \varrho_{=}A(x)\,v\,\mathrm{d}t\,. \tag{8.2}$$

The outflowing mass is

$$\mathrm{d}m_{\text{out}} = \varrho_{=}\left[A(x) + \tfrac{\mathrm{d}A}{\mathrm{d}x}\mathrm{d}x\right]\left(v + \tfrac{\partial v}{\partial x}\mathrm{d}x\right)\mathrm{d}t =$$

$$\varrho_{=}\left[A(x)\,v + A(x)\frac{\partial v}{\partial x}\mathrm{d}x + v\frac{\mathrm{d}A}{\mathrm{d}x}\mathrm{d}x + \overbrace{(\cdots\cdots)}^{2^{\text{nd}}\text{ order differentials}}\right]\mathrm{d}t\,. \tag{8.3}$$

Neglecting the second-order differentials in the sum, we get the mass surplus as follows,

$$\mathrm{d}m_{\Delta} = -\varrho_{=}A(x)\left(\frac{\partial v}{\partial x} + \frac{1}{A(x)}\frac{\partial A}{\partial x}v\right)\mathrm{d}t\,\mathrm{d}x = \overbrace{\left(A(x) + \frac{\mathrm{d}A(x)}{2}\right)}^{\text{average area}}\mathrm{d}x\,\frac{\partial\varrho}{\partial t}\,\mathrm{d}t\,. \tag{8.4}$$

Again neglecting second-order differentials consequently yields

$$-\varrho_{=}\left(\frac{\partial v}{\partial x} + \frac{1}{A(x)}\frac{\mathrm{d}A}{\mathrm{d}x}v\right) = \frac{\partial\varrho}{\partial t}\,, \tag{8.5}$$

which is the *modified continuity equation*.

To get p as the second field quantity instead of ϱ, the state equation (8.1) is used and we arrive at

$$-\left(\frac{\partial v}{\partial x} + \frac{1}{A(x)}\frac{\mathrm{d}A}{\mathrm{d}x}v\right) = \kappa_{=}\frac{\partial p}{\partial t}\,. \tag{8.6}$$

Combining (8.1) and (8.6) leads to *Webster's* equation, which is

$$\frac{\partial^2 p}{\partial x^2} + \left[\frac{1}{A(x)}\frac{\mathrm{d}A}{\mathrm{d}x}\right]\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\,. \tag{8.7}$$

Please note that the term $[1/A(x)]\,(\mathrm{d}A/\mathrm{d}x)$ is identical to $\mathrm{d}[\ln A(x)]/\mathrm{d}x$. For $A(x) = \text{const}$, *Webster*'s equation reduces to the normal one-dimensional wave equation[1].

For a number of analytically defined area functions, *Webster*'s equation can be integrated in closed form. In the following section, we take two of them as examples, namely, conical and exponential horns.

8.2 Conical Horns

For the conical horn – sketched in Fig. 8.3 – the area function is

$$A(x) = A_0\left(\frac{x}{x_0}\right)^2\,. \tag{8.8}$$

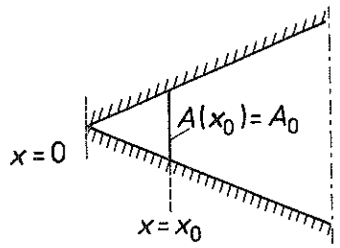

Fig. 8.3. Conical horn

Inserting the area function into *Webster*'s equation results in

$$\frac{\partial^2 p}{\partial x^2} + \frac{2}{x}\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{8.9}$$

whereby we have used

$$\frac{1}{A(x)}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{A_0}{A(x)}\,2\left(\frac{x}{x_0}\right)\frac{1}{x_0} = \frac{2}{x}. \tag{8.10}$$

Equation (8.9) can also be written in the following form as can be proven by differentiating,

$$\frac{\partial^2 (p\,x)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 (p\,x)}{\partial t^2}. \tag{8.11}$$

By inspecting this formula, it becomes obvious that its form corresponds to the one-dimensional wave equation, yet, instead of p, we now have a product $p\,x = g$. The solution is approached in the well known way by

$$\underline{g}(x) = \underline{p}(x)\,x = \underline{g}_{+}\mathrm{e}^{-\mathrm{j}\beta x} + \underline{g}_{-}\mathrm{e}^{+\mathrm{j}\beta x}. \tag{8.12}$$

By restricting ourselves to the forward progressing (outbound) wave, we get the following results for p and v,

$$\underline{p}_{+}(x) = \frac{\underline{g}_{+}}{x}\mathrm{e}^{-\mathrm{j}\beta x} \quad \text{and} \tag{8.13}$$

$$\underline{v}_{+}(x) = \underline{g}_{+}\left[\frac{1}{\varrho_{=}\,c\,x} + \frac{1}{\mathrm{j}\omega\,\varrho_{=}\,x^2}\right]\mathrm{e}^{-\mathrm{j}\beta x}, \tag{8.14}$$

where the solution for the particle velocity, v, has been found via the solution for p by applying *Euler*'s equation (8.1) with $\beta = \omega/c$ as follows,

$$-\underline{g}_{+}(x)\left[-\frac{1}{x^2}\mathrm{e}^{-\mathrm{j}\beta x} - \mathrm{j}\beta\mathrm{e}^{-\mathrm{j}\beta x}\frac{1}{x}\right] = \mathrm{j}\omega\,\varrho_{=}\,\underline{v}. \tag{8.15}$$

[1] *Webster*'s equation for v looks different from that as derived above for p

The sound-field from the conical horn can be divided into a *near field* and a *far field*. The threshold between the two is defined as the position where the magnitudes of the real and imaginary parts of the velocity are just equal, which is at

$$\left|\frac{1}{\varrho_{=}\,c\,x_{\mathrm{ff}}}\right| = \left|\frac{1}{\mathrm{j}\omega\,\varrho_{=}\,x_{\mathrm{ff}}^2}\right|, \tag{8.16}$$

resulting in a far-field distance of

$$x_{\mathrm{ff}} = \frac{\lambda}{2\pi} = \frac{1}{\beta_{\mathrm{ff}}} = \frac{c}{\omega_{\mathrm{ff}}}, \tag{8.17}$$

with $x < \lambda/2\pi$ being the *near field* and $x > \lambda/2\pi$ the *far field.*

The sound pressure, p, decreases to half with a doubling of the distance, x. In other words, the decrease is 6 dB per distance doubling. For v the situation is more complicated. In the near field, v decreases with $1/x^2$ per distance doubling, which is 12 dB decrease per distance doubling. However, in the far field, v behaves like p with 6 dB decrease per distance doubling.

The field impedance $\underline{Z}_{\mathrm{f}}$ of the conical sound field results from dividing (8.13) by (8.14) and is

$$\underline{Z}_{\mathrm{f}}(x) = \frac{\underline{p}_{+}(x)}{\underline{v}_{+}(x)} = \frac{1}{\frac{1}{\varrho_{=}c} + \frac{1}{\mathrm{j}\omega\varrho_{=}x}} = \varrho_{=}\,c\,\frac{\mathrm{j}\frac{2\pi x}{\lambda}}{1+\mathrm{j}\frac{2\pi x}{\lambda}}. \tag{8.18}$$

For this field impedance we can draw a substitute, namely, a long tube with a short branch where a concentrated mass is positioned – depicted in Fig. 8.4. The right panel of the figure shows an electro-acoustic analogy.

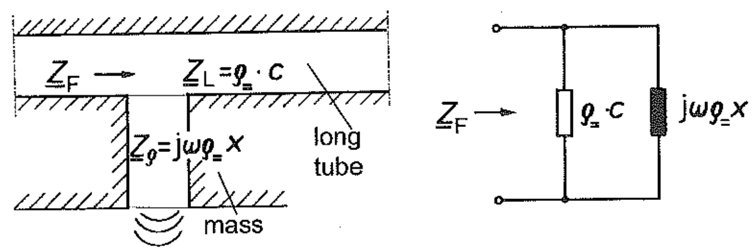

Fig. 8.4. Equivalent circuits for conical horns

The reactive (imaginary) component, $\mathrm{j}\omega\,\varrho_{=}\,x$, is the so-called *co-vibrating medium mass.* This component swings about without transporting active power. The active (real) component $\varrho_{=}\,c$ becomes relatively (not absolutely!) stronger with increasing distance. For $x \gg \lambda/2\pi$, $\underline{Z}_{\mathrm{f}}$ approaches $\varrho_{=}\,c$. Note that $\varrho_{=}\,c$ is the field impedance in a tube with a constant diameter and, thus, the specific field impedance of the medium, Z_{w}.

8.3 Exponential Horns

The area function of the exponential horn – see Fig. 8.5 – is given by

$$A(x) = A_0 e^{2\epsilon x}, \tag{8.19}$$

with $\epsilon > 0$ being the so-called *flare coefficient*.

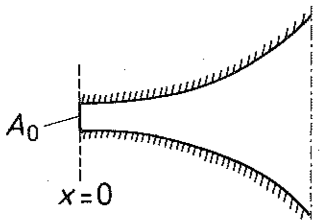

Fig. 8.5. Exponential horn

By differentiation we get

$$\frac{1}{A(x)}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}\,[\ln A(x)]}{\mathrm{d}x} = 2\epsilon \quad \text{and, thus,} \tag{8.20}$$

$$\frac{\partial^2 p}{\partial x^2} + 2\,\epsilon\,\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\,. \tag{8.21}$$

The structure of this equation can most easily be understood by applying complex notation, which leads to

$$\frac{\delta^2 \underline{p}}{\partial x^2} + 2\epsilon\frac{\partial \underline{p}}{\partial x} + \frac{\omega^2}{c^2}\underline{p} = 0\,, \tag{8.22}$$

an equation which recalls the equation of the damped oscillator – see Section 2.3. We confine to the forward progressing wave again and, consequently, try the approach $\underline{p}(x) = e^{\underline{\gamma} x}$ that leads to the characteristic quadratic equation

$$\underline{\gamma}^2 + 2\epsilon\underline{\gamma} + \frac{\omega^2}{c^2} = 0 \tag{8.23}$$

with its two solutions

$$\underline{\gamma}_{1,2} = -\epsilon \pm \sqrt{\epsilon^2 - \frac{\omega^2}{c^2}} = -\epsilon \pm \mathrm{j}\sqrt{\frac{\omega^2}{c^2} - \epsilon^2}\,. \tag{8.24}$$

The complex quantity, $\underline{\gamma}$, is termed *propagation coefficient*, whereby

$$\underline{\gamma} = \alpha + \mathrm{j}\beta\,, \tag{8.25}$$

with α being the *damping coefficient* and β the *phase coefficient*.

The solution of the wave equation, hence, is an exponential function decreasing with x. This is called *spatial damping*. The general solutions for p and v in the forward progressing wave can be formulated as follows,

$$\underline{p}_+(x) = \underline{p}_+(0)\,e^{-\epsilon x}e^{-\mathrm{j}(\sqrt{\frac{\omega^2}{c^2}-\epsilon^2})\,x} = \underline{p}_+(0)\,e^{-\alpha x}e^{-\mathrm{j}\beta x} \quad \text{and} \tag{8.26}$$

$$\underline{v}_+(x) = \frac{\epsilon + \mathrm{j}\sqrt{\frac{\omega^2}{c^2} - \epsilon^2}}{\mathrm{j}\omega\,\varrho_=}\,\underline{p}_+(x)\,. \tag{8.27}$$

Again, the solution for v has been derived from the one for p by applying *Euler*'s equation (8.1).

A prerequisite for wave propagation is that the expression under the square root is positive and, thus, results in a phase coefficient, β. This is the case when $\omega^2/c^2 > \epsilon^2$ and, accordingly, $2\pi/\lambda > \epsilon$ holds. In fact, this is fulfilled above a limiting frequency

$$\omega_\mathrm{l} = \epsilon\, c\,. \tag{8.28}$$

Below ω_l, there is an exponential fade-out as the expression under the root becomes negative, and we then end up with pure damping without wave propagation. Physically, this means that mass is shifted about, but no energy is transported as no sufficient compression takes place. ω_l decreases with decreasing flare coefficient, ϵ. In other words, the slimmer the horn, the lower the limiting frequency.

Please note that the phase velocity, c_ph, in the exponential horn, is different from that in a free plane wave, c, namely,

$$c_\mathrm{ph} = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \epsilon^2}}\,. \tag{8.29}$$

Furthermore, c_ph is frequency-dependent. This effect is called *dispersion* since different frequency components travel with different speed and, thus, the different wave components arrive at the end of the horn at different instances[2].

The so-called *group-delay distortions*, which describe the frequency-dependent delay of the envelope of a transmitted signal, are highest close to the limiting frequency. The group delay , τ_gr, over a wave-traveling distance of l is in our case

$$\tau_\mathrm{gr} = \frac{\mathrm{d}\beta}{\mathrm{d}\omega} = \frac{l}{c\sqrt{1 - (\frac{\omega_\mathrm{l}}{\omega})^2}}\,, \tag{8.30}$$

The field impedance in the exponential horn, $\underline{Z}_\mathrm{f}$, is given by

$$\underline{Z}_\mathrm{f} = \frac{\underline{p}_+}{\underline{v}_+} = \frac{\mathrm{j}\omega\varrho_=}{\epsilon + \mathrm{j}\sqrt{\frac{\omega^2}{c^2} - \epsilon^2}} = \varrho_=\, c\left[\sqrt{1 - (\frac{\omega_\mathrm{l}}{\omega})^2} + \mathrm{j}(\frac{\omega_\mathrm{l}}{\omega})\right]. \tag{8.31}$$

As with the conical horn, $\underline{Z}_\mathrm{f}$ approaches $\varrho_=\, c = Z_\mathrm{w}$ with increasing frequency since we have $\underline{Z}_\mathrm{f} \Rightarrow \varrho_=\, c$ for $\omega \gg \omega_\mathrm{l}$.

[2] This, by the way, contributes to the characteristic sound of horn loudspeakers

8.4 Radiation Impedances and Sound Radiation

The acoustic power that is sent out by an electro-acoustic transducer or any other sound source, is proportional to the real part of the impedance, $r_{\text{rad}} = \text{Re}\{\underline{Z}_{\text{rad}}\}$ that terminates the source at its acoustic output port. Since this impedance is formed by the sound field coupled to the source, we call it *radiation impedance* $\underline{Z}_{\text{rad}}$, and its real part *radiation resistance*. The radiation impedance is a mechanic impedance – refer to Section 4.5 – namely,

$$\underline{Z}_{\text{rad}} = \frac{\underline{F}}{\underline{v}} . \tag{8.32}$$

The radiated power, then, is

$$\overline{P} = \frac{1}{2}\, \text{Re}\{\underline{Z}_{\text{rad}}\}\, |\underline{v}|^2 = \frac{1}{2}\, r_{\text{rad}}\, |\underline{v}|^2 . \tag{8.33}$$

The following relation holds between the field impedance, $\underline{Z}_{\text{f}}$, and the radiation impedance, $\underline{Z}_{\text{rad}}$,

$$\underline{Z}_{\text{rad}} = \int_{\text{A}} \underline{Z}_{\text{f}}\, \text{d}A , \tag{8.34}$$

with A being the effective radiation area.

For transducers that radiate into a horn, the effective area is equal to the area of the mouth of the horn in the optimal case. In the synopsis shown in Fig. 8.6, we assume that the tube/horn is so long that no waveforms are reflected back from the opening, but that the diameter is still small as compared to the wavelength, that is $d \ll \lambda$. This is, of course, an idealized assumption.

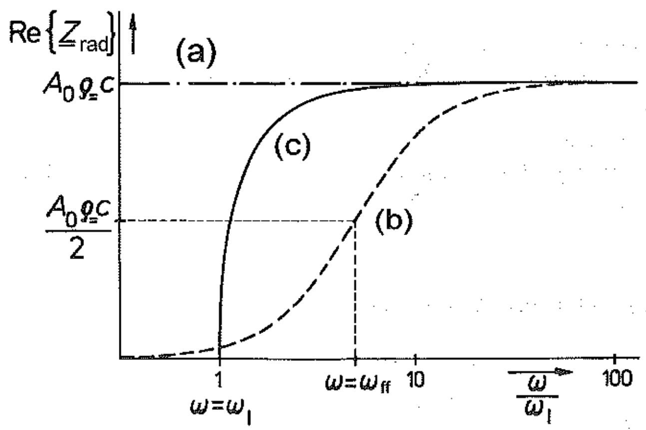

Fig. 8.6. Schematic plot of the radiation resistance of **(a)** a tube, **(b)** a conical horn, and **(c)** an exponential horn. Frequencies normalized to the limiting frequency, ω_{l}, of the exponential horn

For the tube with a constant cross section, we get

$$r_{\text{rad}} = Z_{\text{w}}\, A_0 = \varrho_{=}\, c\, A_0 , \tag{8.35}$$

for the conical horn

$$r_{\text{rad}}\,(A_0) = A_0\, \text{Re}\{\underline{Z}_{\text{f}}\} = A_0\, \varrho_{=}\, c\, \frac{(\frac{\omega}{c}\, x_0)^2}{1 + (\frac{\omega}{c}\, x_0)^2} , \tag{8.36}$$

and for the exponential horn

$$r_{\text{rad}}\,(A_0) = A_0\, \text{Re}\{\underline{Z}_{\text{f}}\} = A_0\, \varrho_{=}\, c\, \sqrt{1 - (\frac{\omega_{\text{l}}}{\omega})^2} . \tag{8.37}$$

For the conical horn, ω_{ff}, which forms the threshold between near- and farfield at a given distance from the mouth, x_1, is independent of the opening angle of the horn. For the exponential horn, however, the limiting frequency, ω_{l} depends on the flare coefficient, ϵ.

The exponential horn is, among all horns that can be described with *Webster*'s equation, the one with the steepest increase of $\text{Re}\{\underline{Z}_{\text{rad}}\} = r_{\text{rad}}$ as a function of frequency. However, by considering the curvature of the waves, one can find even more advantageous forms – for example, spherical-wave horns.

8.5 Steps in the Area Function

We shall now discuss the situation at the position of the step in a tube – shown in Fig. 8.7. Left and right of the step, we have tubes with constant, though different diameters. As already mentioned at the beginning of this chapter, perpendicular modes at this position may be neglected because they cannot propagate as long as $d \ll \lambda$ holds at both sides of the step.

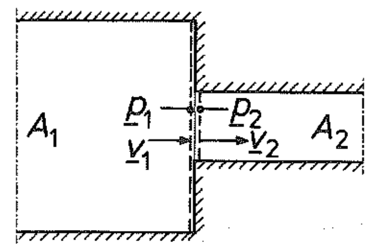

Fig. 8.7. Tube with steps in the area function

This means that, slightly away from the step, we only have axial waves again.

The boundary conditions, thus, are

$$\underline{p}_1 = \underline{p}_2\,, \quad \text{and} \tag{8.38}$$

$$A_1\,\underline{v}_1 = A_2\,\underline{v}_2, \quad \text{which is} \quad \underline{q}_1 = \underline{q}_2\,. \tag{8.39}$$

This means that both quantities are taken as continuous at the step. At the step a reflected wave is created. Accordingly, by combining

$$\underline{p}_{1+} + \underline{p}_{1-} = \underline{p}_{2+} \quad \text{with} \quad A_1\left(\frac{\underline{p}_{1+}}{Z_\mathrm{w}} - \frac{\underline{p}_{1-}}{Z_\mathrm{w}}\right) = \frac{\underline{p}_{2+}}{Z_\mathrm{w}}A_2\,, \tag{8.40}$$

we get a reflectance

$$\underline{r} = \frac{\underline{p}_{1-}}{\underline{p}_{1+}} = \frac{A_1 - A_2}{A_1 + A_2}\,. \tag{8.41}$$

As $\underline{q}$ is continuous at the step, it makes sense to introduce this quantity to deal with stepped-duct problems rather than the particle velocity $\underline{v}$. The transmission-line equations (7.49) and (7.50) can, thus, be rewritten with $\underline{q}$ instead of $\underline{v}$ as follows,

$$\underline{p}(l) = \underline{p}_0\,\cos\beta\,l + \mathrm{j}Z_\mathrm{L}\,\underline{q}_0\,\sin\beta\,l, \quad \text{and} \tag{8.42}$$

$$\underline{q}(l) = \mathrm{j}\frac{\underline{p}_0}{Z_\mathrm{L}}\,\sin\beta\,l + \underline{q}_0\,\cos\beta\,l\,, \tag{8.43}$$

where

$$Z_\mathrm{L} = \frac{Z_\mathrm{w}}{A} = \frac{1}{A}\sqrt{\frac{\varrho_=}{\kappa_=}} = \sqrt{\frac{m'_\mathrm{a}}{n'_\mathrm{a}}}\,, \tag{8.44}$$

is the specific acoustic impedance of the respective tube. m'_a is the acoustic mass per length, the *mass load*, n'_a is the acoustic compliance per length, the *compliance load*.

The two relevant energies now also come out as length-related quantities, namely, kinetic-energy per length,

$$W' = \frac{1}{2}\,m'_\mathrm{a}\,q^2, \tag{8.45}$$

and potential-energy per length,

$$W' = \frac{1}{2}\,n'_\mathrm{a}\,p^2\,. \tag{8.46}$$

The reflectance at the step between two tubes, each with constant cross-section, results as

$$\underline{r} = \frac{Z_{\mathrm{L}_2} - Z_{\mathrm{L}_1}}{Z_{\mathrm{L}_2} + Z_{\mathrm{L}_1}}\,. \tag{8.47}$$

Please note that by taking $\underline{p}$ analogous to $\underline{u}$, and $\underline{q}$ for $\underline{i}$, we see a complete analogy to the electric transmission line where we observe $\underline{i}_1 = \underline{i}_2$ and $\underline{u}_1 = \underline{u}_2$ at steps.

8.6 Stepped Ducts

With $\underline{q}$ as the second wave quantity, we can easily deal with stepped ducts by means of electric analogies. Furthermore, we can include acoustic concentrated elements into our consideration within the same analogue circuits – refer to Section 2.6. This means, in fact, that the theories of analysis and synthesis of electric networks, including transmission lines with and without losses, can be directly used to deal with acoustical problems. For example, acoustic filters with pre-described transfer functions can be designed in this way, including high-pass, low-pass and band-pass filters. This possibility is, for example, exploited in the design of mufflers for car-exhaust systems.

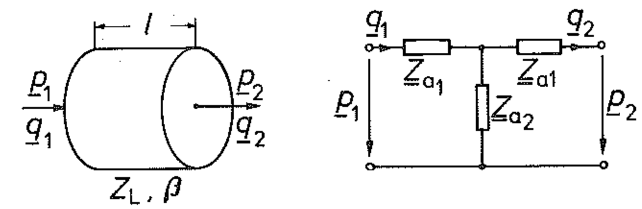

Fig. 8.8. Equivalent circuit for an acoustic tube segment

For the application of this method it is worth recalling that sections of transmission lines and, consequently, acoustic tubes, can be described by T-equivalents as given in the following matrix equation – see Fig. 8.8.

$$\begin{pmatrix}\underline{p}_1\\ \underline{q}_1\end{pmatrix} = \begin{pmatrix}\cos\beta l & \mathrm{j}Z_\mathrm{L}\,\sin\beta l\\ \mathrm{j}\frac{1}{Z_\mathrm{L}}\,\sin\beta l & \cos\beta l\end{pmatrix}\begin{pmatrix}\underline{p}_2\\ \underline{q}_2\end{pmatrix}. \tag{8.48}$$

These are the so-called two-port equations of a tube section, formulated in wave-parameter form. Two-port theory says that the following relations holds,

$$1 + \frac{\underline{Z}_{\mathrm{a}_1}}{\underline{Z}_{\mathrm{a}_2}} = \cos\beta\,l \quad \text{and} \quad \frac{1}{\underline{Z}_{\mathrm{a}_2}} = \mathrm{j}\frac{1}{Z_\mathrm{L}}\,\sin\beta\,l\,, \tag{8.49}$$

further,

$$\underline{Z}_{\mathrm{a}_1} = \mathrm{j}Z_\mathrm{L}\,\tan\frac{\beta\,l}{2} \quad \text{and} \quad \underline{Z}_{\mathrm{a}_2} = -\mathrm{j}Z_\mathrm{L}\,\frac{1}{\sin\beta\,l}\,. \tag{8.50}$$

Please note that because transcendental functions (tan, sin) are involved, the elements can in principle not be realized by concentrated acoustic elements. Yet, for sections of small lengths, that is $l \ll \lambda$, the following approximations apply,

$$\tan\frac{\beta\,l}{2} \approx \frac{\beta\,l}{2} \quad \text{and} \quad \frac{1}{\sin\beta\,l} \approx \frac{1}{\beta\,l}\,. \tag{8.51}$$

Thereupon, with

$$Z_{\mathrm{L}} = \frac{1}{A}\sqrt{\frac{\varrho_{=}}{\kappa_{=}}} \quad \text{and} \quad \beta = \omega\sqrt{\varrho_{=}\,\kappa_{=}}\,, \tag{8.52}$$

we get

$$\underline{Z}_{\mathrm{a}_1} \approx \frac{1}{2}\,\mathrm{j}\omega\,\frac{\varrho_{=}}{A}\,l = \frac{1}{2}\,\mathrm{j}\omega\, m'_{\mathrm{a}}\,l \quad \text{and} \quad \underline{Z}_{\mathrm{a}_2} \approx \frac{1}{\mathrm{j}\omega\,\kappa_{=}\,A\,l} = \frac{1}{\mathrm{j}\omega\, n'_{\mathrm{a}}\,l}\,. \tag{8.53}$$

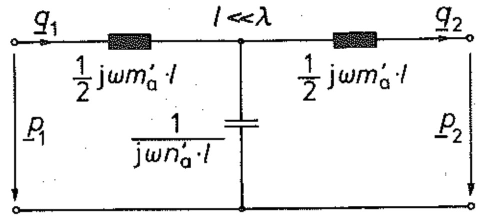

Fig. 8.9. Equivalent circuit for a tube segment

Figure 8.9 shows an equivalent circuit for a short section of a tube. This equivalent circuit is, for example, in use to calculate the transfer function of the human vocal tract or ear canal. The principle is depicted in Fig. 8.10. The higher the attempted accuracy of the calculation, the more sections have to be assumed.

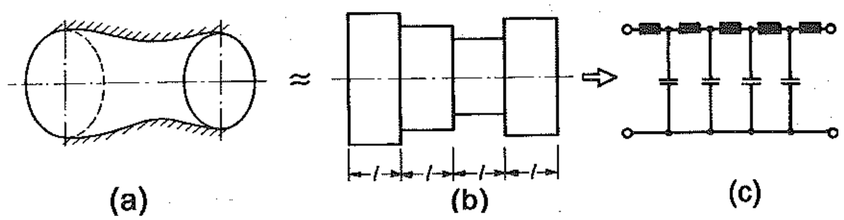

Fig. 8.10. Approximation of a tube with varied cross-sectional area

Finally, in this section, we shall treat the case of a very short narrowing or widening in a tube with otherwise constant cross section. The widening acts like a concentrated, branching spring, Δn, the narrowing like a concentrated serial mass, Δm. This can be figurately conceptualized as follows, taking a widening section with the length $l_2 \ll \lambda$ as example – Fig. 8.11(a).

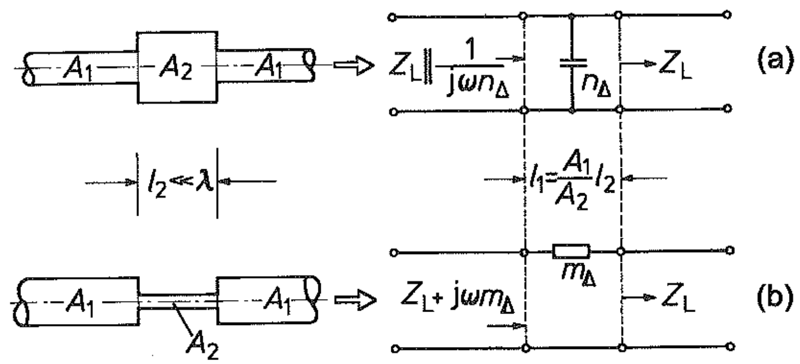

Fig. 8.11. Equivalent circuit for short tube segments, **(a)** short widening in a long constant-diameter tube, **(b)** short narrowing

From (8.53) we learn that

$$m'_{\mathrm{a}_1} = \frac{A_2}{A_1}\,m'_{\mathrm{a}_2} \quad \text{and} \quad n'_{\mathrm{a}_1} = \frac{A_1}{A_2}\,n'_{\mathrm{a}_2}\,. \tag{8.54}$$

Now, the widening section with the length contains the mass

$$m_{\mathrm{a}} = m'_{\mathrm{a}_2}\,l_2\,. \tag{8.55}$$

If this mass were loaded upon the constant-diameter tube, we would need the length

$$l_1 = \frac{A_1}{A_2}\,l_2\,. \tag{8.56}$$

Yet, a section of cross section A_1 and length l_1 would have a compliance of

$$n_{\mathrm{a}} = \frac{A_2}{A_1}\,n'_{\mathrm{a}_2}\,l_1\,. \tag{8.57}$$

What we actually have at the widening section, is, however,

$$n_{\mathrm{a}} = n_{\mathrm{a}_2} l_2 = \left(\frac{A_1}{A_2}\right) n_{\mathrm{a}_1} \left(\frac{A_1}{A_2}\right) l_1 = \left(\frac{A_1}{A_2}\right)^2 n_{\mathrm{a}_1} l_1\,. \tag{8.58}$$

In other words, the widening section acts like a section of the constant-diameter tube that has a cross section of A_1 and a length of

$$l_1 = \frac{A_1}{A_2}\,l_2\,, \tag{8.59}$$

plus an additional parallel spring of

$$n_{\Delta} = \left[\left(\frac{A_2}{A_1}\right)^2 - 1\right] n_{\mathrm{a}}\,. \tag{8.60}$$

For the additional serial mass for a narrowing section, m_{Δ} – see Fig. 8.11(b) – the explanation would run accordingly.