temperature with height also causes refraction, but is less easily observed than wind-induced refraction.

The common misconception that sound travels in straight lines is confuted by one's ability to hear the noise of a road vehicle even when it passes out of view behind an isolated building. The frequency dependence of the diffraction phenomenon accounts for the fact that only the lower-frequency components manage to circumvent the obstacle. By means of the same phenomenon, sound can be heard to emerge from a gap under a heavy closed door between two rooms.

That sound waves transport energy is evidenced by the rattling of the window pane by a passing truck; energy has clearly been transferred to the window. The decay of the climatic final chord of a romantic symphony into silence can only be explained by a conversion of the sound energy produced by the orchestra into some inaudible form. It is, in fact, largely transformed into heat in the clothing of the audience.

The independence of intersecting sound waves has already been seen to give us the ability to understand speech and to enjoy music in enclosed spaces. The remarkable capability of the human auditory system to make sense of the jumble of multiple wave reflections from the enclosing surfaces does the rest.

The fact that sound can be generated by mechanisms other than the vibration of solid bodies is evidenced by the hum of a cooling fan and the roar of a jet engine; you can also try blowing against the tip of your finger. The delicacy of aerodynamic sound generation mechanisms, essential to wind instruments, is typified by the human whistle. Try slowly moving the edge of a piece of horizontally held thin card towards the opening between the lips. You might also like to try to ascertain by what mechanism you change the pitch of your whistle. The 'clack' of colliding snooker balls is caused by their sudden accelerations and decelerations, and not by their resulting vibration, which occurs at frequencies above the audible range. If you don't play snooker (or pool), you might like to observe the sound of pebbles crashing together on a beach lashed by a stormy sea.

These, and many more examples, reveal sound as a fascinating phenomenon that we shall now proceed to analyse in a more scientific, quantitative manner in the next chapter.

## Questions

The fluid is assumed to be air at a pressure of $10^5$ Pa and a temperature of 20°C, unless otherwise stated. Impedance ratios are referred to the characteristic impedance of air under these conditions, unless otherwise stated.

2.1 Write a computer program to determine the sound pressure field generated by a line array of ten impulsive, omnidirectional point sources spaced at equal intervals of 300 mm. Assume that the pressure–time history of the sound field radiated by each source takes the form $p = P_0/r$ for the period $r/c - \Delta t \leq t < r/c$ and $p = -P_0/r$ for the period $r/c < t \leq r/c + \Delta t$, where $r$ is the distance travelled and $\Delta t$ is $10^{-2}$ ms. Evaluate the sound pressure fields in any plane containing the array at intervals of elapsed time of 1 ms up to a total of at least 20 ms. $P_0$ is arbitrary. A colour scale plot of $\log_{10} p$ would be useful.

# 3
# Sound in Fluids

## 3.1 Introduction

This chapter concerns the mechanisms and mathematical expression of sound in fluids. A brief account of the physical properties of fluids, which determine the form of acoustic wave that they support, is followed by a descriptive treatment of the kinematic, dynamic and thermodynamic processes involved. The mathematically based section presents the derivation of various forms of the general equation that governs the behaviour of acoustic waves in fluids, together with some examples of their solutions and interpretations. Suggestions for demonstrations and experiments that assist the understanding of the behaviour of sound waves are provided in Appendix 7.

## 3.2 The physical characteristics of fluids

Although the acoustic behaviour of most commonly encountered materials in the audio-frequency range may be analysed without explicit reference to their molecular nature, it may be helpful to the reader to review briefly the different molecular structures of solids, liquids and gases, in preparation for the introduction of the continuum model. The molecules that form material substances attract each other except where they are in very close proximity, when they exert strong forces of mutual repulsion. Therefore, when molecules approach each other under the influence of the mutually attractive force, they lose potential energy – as does a falling apple. At the point where the interaction force changes from attractive to repulsive, the sum of the potential energies associated with the two forces is a minimum, known as the 'pair dissociation energy'. This state of equilibrium may be disturbed by the impact of other molecules. If the average kinetic energy of the intruder is much less than the dissociation energy it will be captured, and eventually a large conglomerate of bound atoms will form: this is the case in the solid phase of matter. On the other hand, if the average kinetic energy greatly exceeds the dissociation energy, molecules will never 'bond' for any significant time: this is the gaseous phase of matter. Liquids fall in between these two states where 'bonds' are temporarily made and then broken by encounter with molecules of higher than average energy. (This account is loosely based upon that presented in *Three Phases of Matter* (Walton, 1983) – see Bibliography.)

The spacing of molecules in solids is such that the shape of the structure is maintained by strong attractive forces. The molecules simply undergo very small vibrational motions unless they acquire so much energy due to heating that they break free of the attractive forces to form a liquid (or, where supplied with sufficiently high thermal energy, to sublime directly into a vapour). In liquids, the molecules move relative to each other in complex paths under the combined influence of forces of attraction and repulsion, allowing the fluid readily to undergo large changes of geometric form under the action of applied forces, so that they adapt their shape to conform to that of a rigid container. In gases, the average spacing of the molecules is so large that attractive forces are very weak; any individual molecule may translate over a substantial distance before coming sufficiently close to another for the force of repulsion to produce a rapid exchange of momentum, in analogy with the collision of billiards balls. For example, the average distance travelled by a molecule between successive collisions in the air around you is $8 \times 10^{-8}$ m, which is about 25 times the average molecular spacing; so, on average, each molecule passes 24 others between collisions. Gases, unlike liquids, characteristically fully occupy any container. (Some molecules near the free surface of contained liquids temporarily escape to form a co-existing gas-like vapour that occupies the volume of a container not occupied by the liquid.)

The term 'fluid' implies flow. Flow is usually spatially non-uniform in that it entails relative motion of different elements of the medium and frequently involves intermixing of fluid elements. A principal distinction between fluids and solids is that the former cannot resist *steady applied shear forces*, which act so as to 'slide' adjacent layers of material over each other. Liquids and gases are therefore both fluids. Solids react to steady shear forces by undergoing shear distortion, which generates proportional opposing forces, so that a state of static equilibrium is attained. Fluids produce no equivalent reaction to steady shear. However, in common with solids, fluids resist changes of volume occupied by any fixed mass of molecules (volumetric strain); this property is essential to the phenomenon of sound in fluids.

Fluids also exhibit fluid friction, or viscosity, whereby they resist relative 'sliding' motion associated with differential *velocities* of adjacent elements; this acts most noticeably in boundary layers close to bodies moving through fluids. The principal mechanism of viscosity in liquids is intermolecular attraction. Given the freedom of gas molecules, it is somewhat surprising that gases also exhibit viscosity. The principal mechanism is an exchange of mean (time-average) molecular momentum via random molecular transport between adjacent fluid layers moving at different mean velocities. Molecules moving from a fluid element possessing a certain mean velocity into one having lower mean velocity bring with them greater mean momentum than those in the slower element. Satisfaction of conservation of momentum in the absence of external forces requires that the mean momentum of the slower element increases, and vice versa. The effect is to reduce the relative velocities between the elements; the associated rates of change of momentum may be attributed to an internal viscous stress. Fluid viscosity has profound effects within the fibrous and porous materials used as sound absorbers, and in thin tubes. It is also central to the processes of sound generation by turbulent fluid flow.

## 3.3 Molecules and particles

In the air around you, a cube of 1 mm side length contains $2.687 \times 10^{16}$ molecules. For the practical purposes of engineering acoustics it is convenient and scientifically acceptable to model fluids as continuous media. The discrete molecular model is implicitly replaced by a voidless medium of which the properties, state and behaviour at a 'point' are expressed in terms of quantities that are governed by the average state of the multitude of molecules within a 'small' volume containing that point. These quantities are known mathematically as the variables of the model. A region may be considered to be 'small' if the spatial changes of the variables across it may be accurately expressed as the products of the local spatial gradients of the variables and the width of the region. We shall use the term 'element' to express this concept.



The concept of the 'particle' is adopted by fluid dynamicists in describing the kinematic (motional) state of a fluid. This is a fictitious entity that allows us to express the average position, velocity and acceleration vectors of the molecules in a small region surrounding the point of interest. Note carefully that particle velocity (vector) does not relate to the average speed (scalar) of the associated molecules; the square of the latter is characterized by the local temperature of the fluid as a measure of the average molecular kinetic energy. In a fluid that is at rest in a continuum sense (quiescent), the mean vector velocity of the molecules is zero, unlike its temperature. However, the root mean square speed of gaseous molecular motion in any individual direction is very close to the speed of propagation of sound, which is consistent with the concept of the molecule as the acoustic 'messenger'.

## 3.4 Fluid pressure

The principal mechanism of pressure in gases derives from random molecular motion, and the contribution from intermolecular forces of attraction is negligible, whereas they are of comparable effect in a liquid. Here we shall concentrate on gases, leaving a brief discussion of liquids to Section 3.8.

Imagine a very thin rigid sheet suspended within a gas which, in a continuum sense, is at rest. A molecule approaching one surface of the sheet is repulsed by the solid molecules and 'bounces off' the sheet; hence its vector momentum is changed. The sheet is thereby subjected to an impulse equal to this change. Because the individual impulse is so small, and the mean rates of impacts occurring on both sides of the sheet are extremely high and equal, the sheet is subject to zero mean force. The mean rate of change of momentum of the molecules that impact upon unit area of one side of the sheet is defined as the fluid 'pressure': it has the dimensions of force per unit area. If the sheet is removed infinitely slowly, the fluid may be assumed to remain at rest in a continuum sense. Across the former plane of separation there is clearly a symmetry to the exchange of molecules and to exchange of momentum via molecular collision. In terms of the continuum model, the pressures exerted on each other by the formerly separated fluid elements are equal. Since molecules move randomly in all directions with equal probability, the imaginary sheet may be placed in any plane, demonstrating that fluid pressure is not preferentially directed: it is a *scalar* quantity, unlike a force. However, the action of pressure on any surface, whether that of a solid, that of an interface between different fluids, or that of a fluid element, produces a force that is directed *normal* to that surface. Area elements possess both size and spatial orientation and are thus vector quantities: scalar pressure times vector area creates vector force.

## 3.5 Fluid temperature

Temperature is a measure of the average kinetic energy per molecule. Reference to the earlier discussion of dissociation energies and phases of matter qualitatively explains the transition from solid, through liquid, to gas, as temperature is increased by the action of some energy-providing agent. It does not, however, explain the sudden transitions between phases that are undergone by very large conglomerations of molecules. We shall not be concerned further with this incompletely understood phenomenon.

## 3.6 Pressure, density and temperature in sound waves in a gas

We have seen how the concepts of pressure and temperature, as attributed to a gaseous continuum, can be understood in terms of molecular motion. Continuum density is a measure of the average total molecular mass per unit volume of fluid. Molecules in a region of gas that is stationary with respect to some appropriate frame of reference, such as the local surface of the Earth, move in all directions with equal probability. Consequently, their centre of mass is stationary in that respect, even though some molecules may leave, and an equal number enter, the region. The concept of continuum particle displacement implies that the molecules associated with the particle have a non-random average displacement superimposed upon their random displacements, so that their associated mass undergoes displacement: similarly with velocity and acceleration. Sound waves involve time-dependent changes of all these continuum quantities. We shall now study the associated relations between them.

The equilibrium pressures and temperatures of gases, which form components of most systems of interest in engineering acoustics, are such that the gases very closely obey the Equation of State of a Perfect (or Ideal) Gas. This is expressed by

$$P/\rho = RT \tag{3.1}$$

where $P$ is pressure, $T$ is absolute temperature (degree Kelvin), $\rho$ is density and $R$ is a factor that is a function of the type of gas. For air, $R$ is 287 J $kg^{-1}$ $K^{-1}$. *This fundamental relation remains true whatever the process that changes the state of the gas.*

The relation between variations of density about its mean value and associated variations of pressure about its mean value determines the speed of propagation of sound in fluids. Isaac Newton assumed that the temperature of the air remains constant in a sound wave and arrived at a speed which is 16% too low. Equation (3.1) indicates that the implication of his *isothermal* assumption is that sound is a *linear* phenomenon in which pressure is proportional to density. Over a century was to elapse before, in 1816, Pierre Simon, Marquis de Laplace, finally published a derivation of the correct speed, after many others had failed.

The temperature actually rises and falls in concert with pressure and density in a sound wave in a gas; but, at audio frequencies, negligible heat flows between the regions of increased and reduced temperature. These regions are so far apart (half a wavelength in a plane travelling wave) that the temperature gradients are too small to produce significant heat conduction. Thus sound in air is an *adiabatic* process in which the pressure is related to density in the form $P=\alpha\rho^{\gamma}$, where $\alpha$ is a constant and the exponent $\gamma$ is the ratio of specific heats at constant pressure and constant volume, which has the value 1.4 for air. Sound is therefore an essentially *non-linear* phenomenon, as illustrated by Fig. 3.1. However, the fractional changes of density and pressure associated with sound levels tolerable by human beings are so small that the non-linearity has negligible effect, and the slope of the tangent to the curve in Fig. 3.1 at the equilibrium point is a sufficiently accurate measure of the variation of sound pressure with density. (For example, 1 m



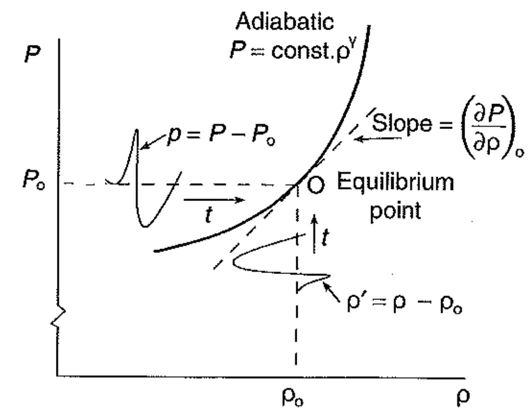


**Fig. 3.1** Adiabatic pressure–density relation.

from the mouth of a typical male speaker, the fractional changes of pressure and density are of the order of $10^{-6}$.) This slope is given by

$$(\partial P/\partial\rho)_0 = (\alpha\gamma\rho^{\gamma-1})_0 = [(\gamma/\rho)\,(\alpha\rho^{\gamma})]_0 = \gamma P_0/\rho_0 = \gamma R T_0 \tag{3.2}$$

in which the subscript 0 indicates the condition of equilibrium, and $P_0$, $\rho_0$ and $T_0$ are the equilibrium pressure, density and temperature, respectively. Note that the partial derivative of pressure with respect to density is used because pressure is a function of other variables. Derivation of an expression for the variation of temperature with density and pressure in terms of these equilibrium values is delegated to the student as an exercise.

Changes of density are associated with changes of volume occupied by a given mass. The preceding relation between sound pressure and density can be expressed as one between sound pressure and volumetric strain. For a fixed mass $M$ occupying mean volume $V_0$, and small changes of volume and density $\delta V$ and $\delta\rho$,

$$M = \rho_0 V_0 = (\rho_0 + \delta\rho)\,(V_0 + \delta V) \tag{3.3}$$

Therefore, correct to first order (products and squares of small quantities neglected),

$$\delta\rho = -\rho_0\,(\delta V/V_0) \tag{3.4a}$$

and

$$\partial\rho/\partial V = -\rho_0/V_0 \tag{3.4b}$$

Using Eqs (3.1) and (3.4),

$$\delta P = P - P_0 = (\partial P/\partial\rho)_0\,\delta\rho = -(\gamma P_0)\,(\delta V/V_0) = -(\gamma R\,\rho_0\,T_0)\,(\delta V/V_0) \tag{3.5}$$

in which $\delta V/V_0$ represents volumetric strain (note the negative sign in Eq. (3.5)). The deviation from equilibrium pressure $P-P_0$ is termed the 'acoustic' or 'sound pressure'; it will henceforth be denoted by $p$ and the associated density deviation will be denoted by $\rho'$. Hence $\partial p/\partial V = -\gamma P_0/V_0$. Fractional changes of pressure, density and absolute temperature in sound waves in air are very small. The value of zero dB in Table 3.1 corresponds to an rms fractional pressure deviation of $2\times10^{-10}$ (see Appendix 6).

Equation (3.5) illustrates the essentially elastic behaviour of ideal gases in response to

**Table 3.1** Examples of typical sound pressure levels

| Source/Environment | Sound pressure level $L_p$ (dB(A)) |
|---|---|
| Launch noise outside rocket payload bay | 160 |
| Heavy artillery at gunners' heads | 140 |
| Threshold of pain | 130 |
| Large jet engine at 30 m; within large symphony orchestra playing fortissimo | 120 |
| 10 m from loudspeakers at rock concert; 1 m from pneumatic chipping hammer | 110 |
| Inside a textile factory; in an old-fashioned underground train | 90 |
| Shouted male voice at 1 m; dense, accelerating road traffic at kerbside; inside jet airliner at take-off | 80 |
| Dense, free-flowing road traffic at 3 m from kerbside | 70 |
| Busy restaurant; two-person conversation | 60 |
| Average commercial office | 50 |
| Residential, urban neighbourhood, far from main roads, at night; library with no air-conditioning | 40 |
| Theatre with full audience just before curtain up | 30 |
| Empty recording studio; empty symphony hall | 20 |
| Male human breathing at 3 m | 10 |
| Average threshold of hearing of 1 kHz tone of normally hearing young persons | *c*. 0 |

small volumetric strain; the resulting stress (acoustic pressure) is linearly proportional to strain, in accordance with Hooke's law. The constant of proportionality, $\gamma P_0$ or $\gamma R \rho_0 T_0$, is the adiabatic bulk modulus of the gas. In air at sea level its value is approximately $1.4 \times 10^5$ N m$^{-2}$ (Pa). The inverse of the bulk modulus is termed 'compressibility'. In accordance with our previous description of the nature of fluids, we note that it is only volume strain that can generate a reactive stress; changes of shape cannot be resisted.

At this point it is timely to try to draw the distinction between acoustic and non-acoustic pressures. This task cannot be accomplished with complete rigour in this introductory textbook; acousticians have argued about it for 50 years without reaching a complete consensus. Instead, a few examples will be presented in a qualitative manner, which it is hoped will not enrage the cognoscenti. It is well known that if you speak closely into a microphone that is not fitted with a windscreen, an unpleasant 'pop' noise will be superimposed on the recorded voice sound. This will not be apparent to someone listening to you speak 'live' as you make the recording. The microphone is recording some pressure fluctuations that are not associated with sound transmitted to the 'live' listener's ear. These are non-acoustic pressure fluctuations associated with unsteady fluid motion in the airstream leaving your mouth. It will be present if you gently blow on the microphone, producing turbulent flow, even though little live sound can be heard. Both the voice sound and the turbulent flow contain pressure gradients that produce fluid particle accelerations, but the natures of the two flow fields are clearly different: one generates a disturbance propagating at the speed of sound; the other is localized and propagates at the local flow speed. The first is an acoustic field; the other is not. In Chapters 6 and 10 we shall encounter acoustic field components close to sources of sound that do not propagate away from the source; but even here the acoustic relation between pressure and density fluctuations, derived in the previous section, holds good.



In the low-speed turbulence of your breath, the density fluctuations are negligible, and the pressure fluctuations are predominantly associated with the momentum fluctuations of an effectively incompressible fluid. These are the origin of the low-frequency noise that you hear when the wind blows past your ears – so-called 'pseudo sound'.

## 3.7 Particle motion

*Kinematics* concerns the geometry of motion without regard to the causes of motion. *Dynamics* concerns the forces that cause motion and their effects. The kinematic state of a fluid at any instant of time is represented in terms of the instantaneous spatial distribution of vector velocities of the particles of fluid. In terms of classical mechanics, which may be assumed to apply to all the systems of interest in engineering acoustics, the rates of change of particle velocities are related to the total forces acting on them in accordance with Newton's second law of motion. Although viscous forces significantly affect fluid motion very close to solid surfaces, and also dissipate sound energy into heat during sound propagation, the general behaviour of sound fields in both gases and liquids may be analysed with sufficient precision for most practical purposes by assuming them to be inviscid (lacking viscosity). The effects of viscosity are described and analysed in Chapter 7.

As a consequence of this assumption, together with the assumption of the absence of electromagnetic forces, the only remaining *internal* forces acting within a fluid to cause particle accelerations are those due to spatial gradients of pressure. *External* forces can be applied by gravity and by contiguous solid surfaces, such as that of a vibrating loudspeaker cone. Gravitational forces play little direct part in controlling acoustic motion in fluids, although they do control the spatial variations of mean pressure and density in all fluids and also influence the relatively slow variations associated with thermal convection.

## 3.8 Sound in liquids

We have already had a brief look at the differences between gases and liquids in terms of molecular structure. In spite of these differences, audio-frequency sound behaves similarly in *homogeneous* gases and liquids in that its existence and propagation speed depend upon the interaction between inertia and elastic stresses produced by volumetric strain. Liquids are obviously less compressible than gases because of the relative closeness of their molecules and the resulting influence of intermolecular repulsion. The bulk modulus depends principally upon the type of liquid, hydrostatic pressure, temperature and, in the case of sea water, salinity. The difference between the adiabatic and isothermal bulk moduli is generally very small, being less than 1%.

In the case of water, the variation of speed of sound with hydrostatic pressure at a fixed temperature is nearly linear, but at fixed pressure it rises to a maximum and then falls as temperature is increased. The presence of salt slightly increases the sound speed in water. An empirical expression for the sound speed in sea water in the temperature range 0–20°C and pressures between $10^5$ and $10^7$ Pa was developed by Wilson [3.1] as $c = 1490 + 3.6\,\Delta T + 1.6 \times 10^{-6}\,P_s + 1.3\,\Delta S$ m s$^{-1}$, where $\Delta T = T(°\mathrm{K}) - 283.16$, $p$ is the absolute static pressure in Pa and $\Delta S = S - 35$, where $S$ is the salinity in grams of salt per kilogram of water.

The large difference between the compressibility of gases and liquids allows the presence of very small proportions of gas in a liquid to have a profound effect on the speed of sound and also on the attenuation of sound waves through the processes of scattering and absorption. The relatively very large compressibility of small bubbles of gas within a liquid relieves the stresses that would otherwise be produced by volumetric strain: the liquid can intrude upon the gas volumes without inducing significant pressure. The effective bulk modulus is therefore greatly reduced but the mean fluid density is little changed.

Bubbles of gas resident in liquids act as resonators, the stiffness being supplied by the gas and the inertia being supplied by the locally surrounding liquid. The resonance frequency is inversely proportional to bubble diameter. A commonly observed natural phenomenon that results from the transient response (ringing) of bubbles of many different sizes is that of the 'babbling' of a brook.

## 3.9 Mathematical models of sound waves

### 3.9.1 The plane sound wave equation

Sound waves exist in the four dimensions of space and time. The essence of mechanical wave motion is that spatial and temporal variations of the physical quantities involved are linked. In the case of acoustic waves this linking is via thermodynamic, kinematic and dynamic relations, some of which have been explained in earlier sections of this chapter. In certain cases, sound waves take a particularly simple form in that the wavefronts are plane. This means that each acoustic quantity is uniform over any plane surface normal to the direction of propagation. As time progresses, the values of each quantity in any plane vary synchronously according to the time dependence of the sound-generating mechanism.

The simplest practical example is that of a sound field that is generated by a sliding rigid piston at one end of a rigid tube of uniform cross-section that is terminated by a non-reflective (anechoic) termination at the other end. Those familiar with fluid dynamics will immediately object that the particle motion cannot be completely uniform over the entire cross-section of the tube because of the presence of a boundary layer at the tube surface in which the particle motion is constrained to be zero at the wall. Thus we are forced to introduce an assumption into the model that the fluid lacks viscosity (that is to say it is *inviscid*). Analysis of the propagation of sound in a viscous fluid in a tube, presented in Chapter 7, demonstrates that the viscous boundary layer only influences sound propagation to a significant extent in tubes of very small diameter, such as capillary tubes. It is, however, responsible for dissipating sound energy into heat to a small extent in all cases where sound waves exist in fluids bounded by rigid surfaces. Viscosity also acts to produce weak attenuation in all propagating sound waves.

The inviscid assumption greatly simplifies the analysis of sound fields, and it can be justified here by the fact that its neglect produces insignificant error in the analysis of many problems of practical engineering interest. It must, however, be accounted for in the models of sound absorption mechanisms and materials that are presented in Chapter 7. Additional assumptions about the nature of fluids made extensively throughout this book are that they possess the same properties everywhere (they are *homogeneous*), that these properties are direction independent (they are *isotropic*), and that gases satisfy the Perfect Gas Law, as expressed by Eq. (3.1). Any deviation from these assumptions will be explicitly signalled. We shall also assume that the acoustic disturbances are sufficiently small for the fluid to behave as a linear elastic medium. This would not be true, for example, in the exhaust pipes of internal combustion systems, which also present analytical difficulties because the gas is not at uniform temperature and flows at high speed.



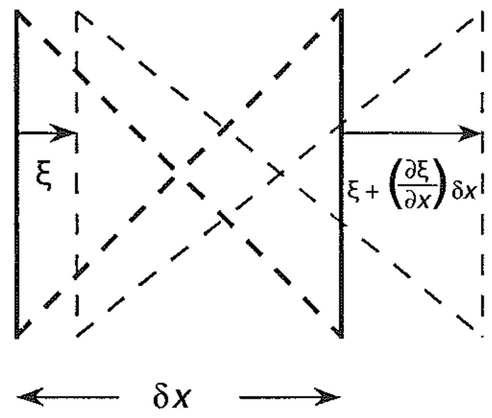


**Fig. 3.2** Plane wave strain.

Having established the assumptions of our model, we may now return to the plane wave in a tube. In the absence of viscous shear stresses, the only remaining internal forces that can accelerate fluid particles result from spatial variations of sound pressure. Since sound pressure is proportional to volumetric strain (Eq. (3.5)), we must suppose that spatial variation of strain is an essential feature of sound waves. Hence we begin our analysis with a graphical representation of strain (Fig. 3.2), which, by the nature of plane waves, is a function of only one space variable: the shape of the tube cross-section is thus immaterial.

The left-hand face of a fluid element of unstrained length $\delta x$ is assumed to undergo a displacement $\xi$ due to some acoustic disturbance. Strain is introduced by assuming the right-hand face to be displaced by a different amount $\xi + \delta\xi$. The differential displacement $\delta\xi$ may be expressed as $(\partial\xi/\partial x)\,\delta x$; the higher-order terms in the Taylor expansion $\xi(x + \partial x) = \xi(x) + (\partial\xi/\partial x)\ \delta x + (\partial^2\xi/\partial x^2)\ (\delta x^2/2) + \ldots$ are neglected in accordance with our previous definition of a 'small' element. The partial derivative is employed because $\xi$ will also be a function of time. Hence the volumetric strain is

$$\delta V/V_0 = S[\xi + (\partial\xi/\partial x)\ \delta x - \xi]/S\delta x = \partial\xi/\partial x \tag{3.6}$$

where $S$ is the cross-sectional area of the tube. The associated acoustic pressure in a gas is given by Eq. (3.5) as

$$p = -\gamma P_0(\partial\xi/\partial x) \tag{3.7}$$

In a liquid, $\gamma P_0$ would be replaced by the relevant bulk modulus.

If the strain $\partial\xi/\partial x$ were uniform over the length of the tube, so would be the pressure, and no wave motion would exist because each fluid element would be in static

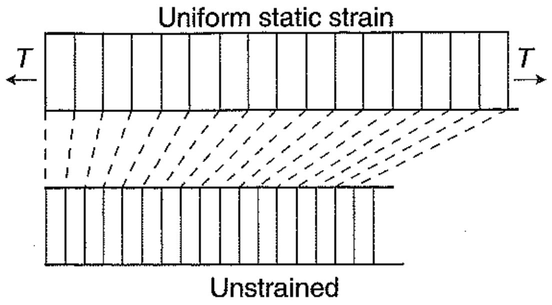


**Fig. 3.3** Uniform strain.

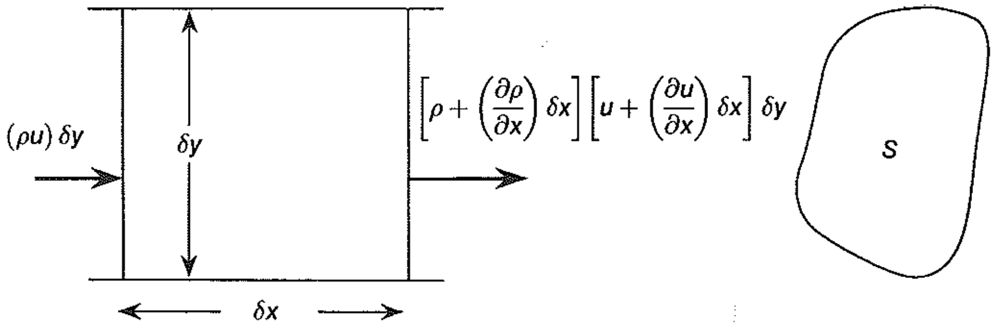


**Fig. 3.4** Mass flux through a control volume.

equilibrium. The equivalent expression for a uniform solid rod under static tension is $\sigma = E(\partial\xi/\partial x)$, where $\sigma$ is the normal stress and $E$ is the elastic modulus (Fig. 3.3). However, there is a vital difference. The walls of the tube prevent lateral displacement of the fluid, whereas the stress-free boundary of the rod allows lateral strain to occur: this is the Poisson effect. It can be clearly observed when a rubber band is stretched.

An alternative approach to the derivation of Eq. (3.7), which is more readily extended to more than one dimension, is to define a 'control volume' that is fixed with respect to the frame of reference relative to which fluid motion is defined. Figure 3.4 shows the rates of flow of mass (flux) through the two faces of the control volume. In accordance with the principle of conservation of mass the instantaneous rate of increase of mass contained in the volume must equal the instantaneous difference between the rates of mass flow into and out of the volume. Thus

$$S\,\delta x(\partial\rho/\partial t) = S\left[\rho u - \left(\rho + \frac{\partial\rho}{\partial x}\delta x\right)\left(u + \frac{\partial u}{\partial x}\delta x\right)\right] \tag{3.8}$$

where $u = \partial\xi/\partial t$ is the particle velocity, and $\rho = \rho_0 + \rho'$ as defined below Eq. (3.5), so that $\partial\rho/\partial x = \partial\rho'/\partial x$ and $\partial\rho/\partial t = \partial\rho'/\partial t$.

Linearization of this equation by the neglect of second-order quantities yields

$$\partial\rho'/\partial t = -\rho_0(\partial u/\partial x) \tag{3.9}$$



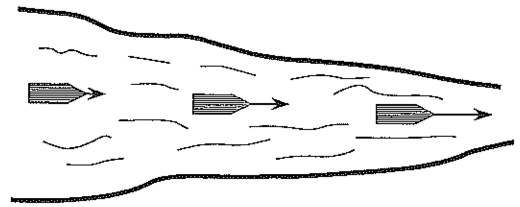

**Fig. 3.5** Acceleration of a boat travelling down a narrowing river.

Now, according to Eqs (3.4) and (3.5), $\rho'/\rho_0 = p/\gamma P_0$, so that Eq. (3.9) can be written

$$\partial p/\partial t = -\gamma P_0(\partial u/\partial x) \tag{3.10}$$

which is the time derivative of Eq. (3.7).

As already mentioned, it is spatial gradients of pressure that cause the accelerations of fluid elements that are essential to wave motion. We now appeal to Newton's Second Law of Motion (N2LM) to relate motions to forces. The mathematical expression of particle acceleration in fluid flow (for sound is a flow phenomenon) is not so simple as for solids because of the phenomenon of particle convection (transport).

Physical understanding may be aided by considering the motion of a small boat floating along a narrowing stream (Fig. 3.5). The flow at any one point is time independent; that is to say the flow is *steady*. But the boat accelerates as it is carried downstream into progressively faster flowing water: this is the *convective* contribution to acceleration that applies even in steady flow. It is expressed mathematically by dividing the velocity change as it moves from $x$ to $x + \delta x$ by the time to traverse distance $\delta x$:

$$a_c = \left[\frac{(u + \frac{\partial u}{\partial x}\delta x - u)}{\delta t}\right] = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t} = u\frac{\partial u}{\partial x} \tag{3.11}$$

where $\delta x/\delta t \to u$ as $\delta t \to 0$. If, however, a sluice gate is suddenly opened upstream of the boat, the flow speed at any point will vary with time; the flow is *unsteady* and $a_t = \partial u/\partial t$. Under these circumstances the boat's acceleration will be a function of both position and time. The total acceleration is then expressed as the sum of two independent contributions:

$$a = a_c + a_t = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \tag{3.12}$$

In sound waves in otherwise quiescent fluids, at amplitudes small enough to satisfy the assumption of linearity, the ratio of the second term to the first term in Eq. (3.12) is of the order of the ratio of the speed of sound to the particle speed. Since particle speeds are typically of the order of $10^{-3}$ m s$^{-1}$, the first term can safely be neglected. However, it may not be neglected in models of turbulent fluid dynamic noise sources such as jet engine exhausts or in the analysis of sound propagation in fluids undergoing net transport (mean flow).

The net force in the $x$-direction on the fluid in the control volume of Fig. 3.4 is produced by the difference of pressures at $x$ and $x + \delta x$ (Fig. 3.6). All the internal forces between the particles in the control volume sum to zero by virtue of Newton's Third Law of Motion (N3LM).

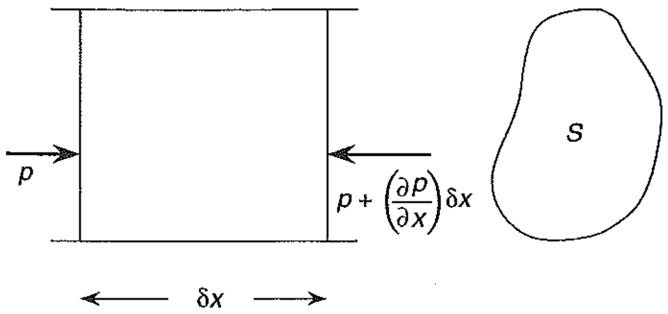


**Fig. 3.6** Fluid element subject to a pressure gradient.

Thus

$$F = S\left[p - \left(p + \frac{\partial p}{\partial x}\delta x\right)\right] = -S\frac{\partial p}{\partial x}\delta x \tag{3.13}$$

The linearized form of N2LM is therefore

$$S\,\delta x\,\rho_0(\partial u/\partial t) = -S\,\delta x(\partial p/\partial x)$$

or

$$\rho_0(\partial u/\partial t) = -\partial p/\partial x \tag{3.14}$$

in which $\rho$ has been replaced by $\rho_0$.

Differentiation of Eq. (3.14) with respect to $x$ yields

$$\rho_0\frac{\partial^2 u}{\partial t\,\partial x} = -\frac{\partial^2 p}{\partial x^2} \tag{3.15}$$

and differentiation of Eq. (3.10) with respect to $t$ yields

$$\gamma P_0\frac{\partial^2 u}{\partial t\,\partial x} = -\frac{\partial^2 p}{\partial t^2} \tag{3.16}$$

Hence,

$$\frac{\partial^2 p}{\partial x^2} = \left(\frac{\rho_0}{\gamma P_0}\right)\frac{\partial^2 p}{\partial t^2} \tag{3.17}$$

which is the plane acoustic wave equation in sound pressure. Density and temperature fluctuations are linearly related to $p$ and hence satisfy the same equation, as do particle displacement, velocity and acceleration.

### 3.9.2 Solutions of the plane wave equation

Equation (3.17) has been derived without reference to any specific sound-generating mechanism; solutions therefore represent all *physically possible* forms of plane sound fields. To use a phrase loathed by students, 'it can be shown' that the equation has the following generation solution:

$$p(x,t) = f[(\gamma P_0/\rho_0)^{1/2}\,t - x] + g\,[(\gamma P_0/\rho_0)^{1/2}\,t + x] \tag{3.18}$$

where $f$ and $g$ are arbitrary functions of their arguments that are determined by the kinematic or dynamic conditions imposed upon the fluid at its boundaries. I justify 'pulling this rabbit out of a hat' by the fact that the proof is involved and adds little to



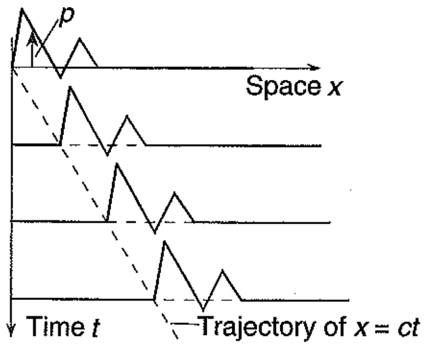


**Fig. 3.7** Illustration of wave speed.

understanding. However, if you wish to indulge in a little masochism, assume a solution to Eq. (3.17) of the form $p = f(u)$, where $u$ is some function of $\gamma P_0/\rho_0$, $x$ and $t$. The less demanding reader will be satisfied by the back substitution of $f$ and $g$ into Eq. (3.17). In the case of an anechoically terminated tube, only one of the two functions exists and the wave is said to be 'travelling' or 'progressive'.

It is clear that, whatever its form, $f$ is constant if the observation point $x$ and time $t$ are related by $[(\gamma P_0/\rho_0)^{1/2}\,t - x]$ = constant, which implies that if the observer travels at speed $(\gamma P_0/\rho_0)^{1/2}$ in the $x$-direction, the observed sound pressure will not change with time. This demonstrates that $(\gamma P_0/\rho_0)^{1/2}$ is the acoustic wave speed, or speed of sound, as illustrated by Fig. 3.7. The function $g$ clearly represents a wave travelling in the negative-$x$ direction. The speed of sound is conventionally represented by the symbol $c$: Eq. (3.1) shows that $c$ is equal to $(\gamma RT_0)^{1/2}$ and is therefore a function only of absolute temperature for a specific gas (given values of $\gamma$ and $R$). Equation (3.18) can now be written

$$p(x,t) = f(ct - x) + g(ct + x) \tag{3.19}$$

From Eqs (3.4) and (3.5) we may now write $p = c^2\rho'$, which is true everywhere in a linear sound field. The adiabatic bulk modulus $\gamma P_0$ equals $\rho_0 c^2$.

### 3.9.3 Harmonic plane waves: sound pressure

Equation (3.19) applies for any form of time-dependence imposed by a source of sound. As explained in Appendix 2, harmonic (single frequency, pure tone) wave behaviour is of fundamental importance, particularly in an analytical sense, since any form of time dependence can be constructed from, and analysed into, a set of harmonic functions. The most convenient form of mathematical expression of harmonic time dependence is the 'complex exponential representation', which is fully explained in Appendix 1. A time-harmonic plane sound pressure field is represented by the expression $p(x,t) = \tilde{p}(x)\exp(j\omega t)$, in which $\omega$ is the angular frequency and $\tilde{p}(x)$ represents the spatial distribution of the complex amplitude pressure, yet to be determined. Introducing this expression into Eq. (3.17) yields the one-dimensional form of the Helmholtz equation

$$\mathrm{d}^2\tilde{p}(x)/\mathrm{d}x^2 + (\omega/c)^2\tilde{p}(x) = 0 \tag{3.20}$$

which has converted the linear, second-order, partial differential equation into a linear, second-order, ordinary differential equation, of which the standard trial function takes the form $\tilde{A}\exp(\lambda x)$. $\tilde{A}$ is the complex amplitude of pressure at $x = 0$. Substituting the trial function into Eq. (3.20) gives

$$(\lambda^2 + k^2)\tilde{p}(x) = 0 \tag{3.21}$$

in which $k = \omega/c$ is the *wavenumber*, which represents spatial frequency, as explained in Appendix 3. Hence the non-trivial (physically meaningful) solution is

$$\lambda = (-k^2)^{1/2} = \pm jk \tag{3.22}$$

There are two solutions because the differential equation is of second order. The complete solution is

$$p(x,t) = \tilde{A}\exp[j(\omega t - kx)] + \tilde{B}\exp[j(\omega t + kx)] \tag{3.23}$$

The exponents can be made compatible with the arguments of $f$ and $g$ in Eq. (3.19) by writing $\omega t \pm kx = k(ct \pm x)$, giving

$$p(x,t) = \tilde{A}\exp[(jk)(ct - x)] + \tilde{B}\exp[(jk)(ct + x)] \tag{3.24}$$

The presence of the negative sign in the exponent of the first term indicates that it represents the positive-going wave and the positive sign indicates a negative-going wave. (Note: physicists and mathematicians generally employ the $\exp(-j\omega t)$ convention, in which case the significance of the signs reverses.) Appendix 1 introduces the graphical 'phasor' representation of a harmonic function. Each of the terms in Eq. (3.23) is harmonic in both time and space. In the complex plane, the phase of the pressure may be visualized by multiplying the complex amplitude $\tilde{A}$, which is time independent, by an anticlockwise rotating unit phasor $\exp(j\omega t)$, representing time dependence, and by another representing space dependence, which takes the form of a clockwise rotating phasor $\exp(-jkx)$ for the positive-going wave and an anticlockwise rotating phasor $\exp(jkx)$ for the negative-going wave.

Consider the pressure variation in time at a *fixed position* $x = 0$, illustrated by Fig. 3.8(a). In the absence of specified boundary conditions, the complex amplitudes $\tilde{A}$ and $\tilde{B}$ are arbitrarily represented. As *time* progresses, *both* phasors rotate in an *anticlockwise* direction at speed $\omega$, as does the phasor representing the sum of the two waves. The projection of the resultant phasor on the real axis, which represents the physical pressure, describes a harmonic oscillation.

Now we *fix the time* at the initial value $t = 0$ and move the observation point in space in the *positive-x* direction. Figure. 3.8(b) shows that the phasor representing the space dependence of the positive-going wave rotates by $kx$ in the clockwise direction (because of the negative sign), while that representing the negative-going wave rotates by $kx$ in the anticlockwise direction. The resultant phasors have different magnitudes and different phases from that in the position $x = 0$. Now we allow time to progress and this resultant phasor rotates at speed $\omega$ in an anticlockwise direction, the projection on the real axis describing a harmonic oscillation of different phase and amplitude from that observed at the position $x = 0$. We can now combine these temporal and spatial variations on a two-dimensional plot. First, we represent only the positive-going wave as shown in Fig. 3.8(c), which reveals why $c$ is known as the 'phase speed' of the wave: the phase is constant for an observer travelling at this speed in the positive-$x$ direction. Note that the physical amplitude is independent of position $x$, but the phase of the pressure varies linearly with $x$. In Fig. 3.8(d), we represent the sum of oppositely directed waves. Note



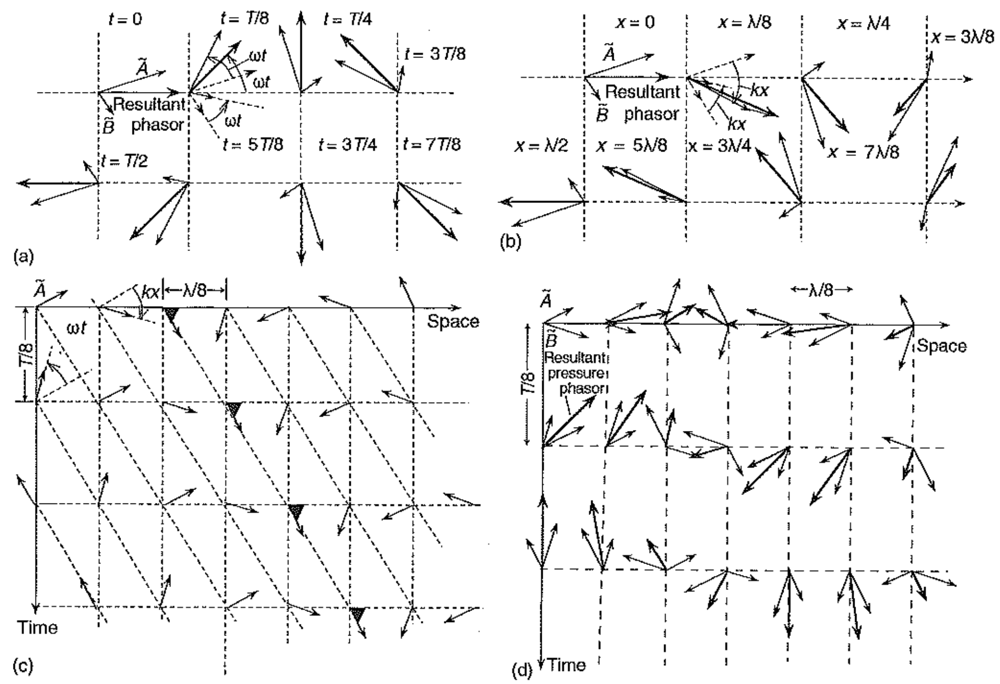


**Fig. 3.8** Phasor representations of pressure in a harmonic field. (a) Variation in time at a fixed position in an interference field formed by oppositely directed plane waves. (b) Variation in space at a fixed time in the interference field. (c) Variation in space and time in a plane progressive wave, and (d) in the interference field.

that *both* the physical amplitude and phase vary with $x$ in Figs 3(b) and 3(d). This is the result of interference – not of interaction.

Equation (3.23) clearly satisfies the principle of linear superposition: the sound pressures of each wave simply add to produce the total pressure. The resulting interference can be most easily seen in the expression for the time-averaged (mean) square pressure. Appendix 1 presents a simple (and almost foolproof) short cut to deriving an expression for the mean square value of any physical quantity that has harmonic time dependence. If $y = \tilde{X}\exp(j\omega t)$, the mean square value of $y$ is given by $\frac{1}{2}\tilde{X}\tilde{X}^*$, in which the asterisk indicates the complex conjugate. The time exponent is therefore extracted from Eq. (3.23) to give the $x$-dependent complex amplitude of pressure, and its complex conjugate, as

$$\tilde{p}(x) = \tilde{A}\exp(-jkx) + \tilde{B}\exp(jkx) \tag{3.25a}$$

and

$$\tilde{p}^*(x) = \tilde{A}^*\exp(jkx) + \tilde{B}^*\exp(-jkx) \tag{3.25b}$$

Hence, the mean square pressure is given by

$$\overline{p^2(x)} = \tfrac{1}{2}\tilde{p}(x)\tilde{p}^*(x) = \tfrac{1}{2}[|\tilde{A}|^2 + |\tilde{B}|^2 + \tilde{A}^*\tilde{B}\exp(2jkx) + \tilde{A}\tilde{B}^*\exp(-2jkx)] \tag{3.26}$$

in which the modulus sign $|\tilde{X}|$ denotes the magnitude of complex amplitude $\tilde{X}$: in other words, the real physical amplitude. (I adopt this apparently unwieldy notation because I

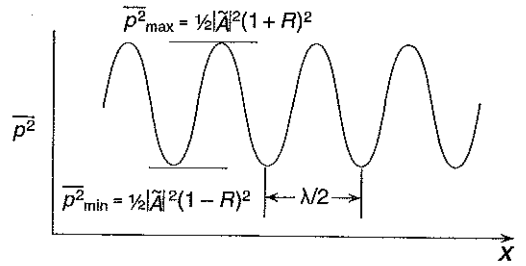


**Fig. 3.9** Spatial distribution of mean square pressure in an interference field formed by oppositely directed, harmonic plane waves: $R = |\tilde{B}/\tilde{A}|$.

have found that students frequently confuse squared amplitudes and mean square values, leading to an error of a factor of two.)

By setting $\tilde{A}$ to $a + jb$ and $\tilde{B}$ to $c + jd$, you may show that the mean square pressure is real, positive and independent of time, which it must be. The first two terms in Eq. (3.26) are independent of $x$, but the third and fourth terms vary as $\cos 2kx$ and $\sin 2kx$. Interference produces maxima and minima of mean square pressure, separated by one-quarter wavelength, as illustrated by Fig. 3.9. The pattern of mean square pressure is *stationary* in space. But, as we have seen, this does not mean that the phase of the pressure is the same at all positions. If the complex wave amplitudes $\tilde{A}$ and $\tilde{B}$ were equal, the maximum and minimum values of $\overline{p^2(x)}$ would be $2|\tilde{A}|^2$ and zero, respectively. In this case, the wave is a pure 'standing wave' and the phase changes by $\pi$ at spatial intervals of half a wavelength. Pure standing waves are rarely generated in practice because they involve no mean energy transport, and energy must travel from a source to regions where it is dissipated into heat. However, the concept of the standing wave is useful because it forms the basis of the modal representation of sound fields in ducts and enclosures, as explained in Chapters 8 and 9. In this case, energy dissipation is accounted for by introducing the concept of modal damping.

*Author's advice*: It is absolutely vital that students acquire confidence in manipulating complex algebraic expressions of harmonic sound fields. Failure so to do will seriously impede progress in developing analytical dexterity and physical comprehension.

### 3.9.4 Plane waves: particle velocity

It is not sufficient to restrict the study of sound fields in fluids to the consideration of the sound pressure alone. It is necessary also to determine the kinematic acoustic behaviour of fluids in order to analyse and understand the processes of acoustic energy transmission and absorption (dissipation), interaction with solid materials, and radiation from vibrating surfaces, among others. The general relation between pressure and fluid motion is expressed by the equation of conservation of momentum (Eq. (3.14)). In the special case of progressive plane wave fields, the general solutions for particle displacement in positive- and negative-going waves take the same form as those for pressure, to which it is linearly related, so that we may express particle displacement in a positive-going wave as $\xi^+(x, t) = h(ct - x)$. Differentiation with respect to time gives the particle velocity

$$u^+(x, t) = \partial\xi^+/\partial t = ch' \tag{3.27}$$



where the prime indicates differentiation of the function with respect to its argument. Differentiation with respect to $x$ gives an expression for the pressure from Eq. (3.7):

$$p^+(x, t) = \rho_0 c^2 h' \tag{3.28}$$

Equations (3.27) and (3.28) give

$$p^+(x, t)/u^+(x, t) = \rho_0 c \tag{3.29a}$$

for positive-going waves. A similar analysis gives

$$p^-(x, t)/u^-(x, t) = -\rho_0 c \tag{3.29b}$$

for negative-going waves.

The quantity $\rho_0 c$, which has the dimensions of pressure/velocity ($ML^{-2}T^{-1}$) and units of kg $m^{-2}$ $s^{-1}$, is a special form of an impedance (see Chapter 4). Since the acoustic properties of a fluid are completely characterized by the mean density $\rho_0$ and the speed of sound $c$, it is thus known as the 'characteristic specific acoustic impedance'. Its unit is named the 'rayl' (after Lord Rayleigh). The presence of the minus sign in Eq. (3.29b), which is because particle velocity is a vector, is explained by Fig. 3.10. The particle velocity field associated with the general plane wave interference field analysed in the preceding section is given by

$$u(x, t) = (1/\rho_0 c)\,[f(ct - x) - g(ct + x)] \tag{3.30}$$

and the corresponding harmonic form is

$$u(x, t) = (1/\rho_0 c)\,[\tilde{A}\exp(-jkx) - \tilde{B}\exp(jkx)]\exp(j\omega t) \tag{3.31}$$

It is left as an exercise for the student to demonstrate that the mean square particle velocity exhibits a similar form of stationary pattern to that of the pressure, but with positions of maxima and minima interchanged.

### 3.9.5 The wave equation in three dimensions

The derivation of the wave equation in three space dimensions is simply an extension of that of the plane wave equation. Figure 3.11 shows a rectangular parallelepiped control

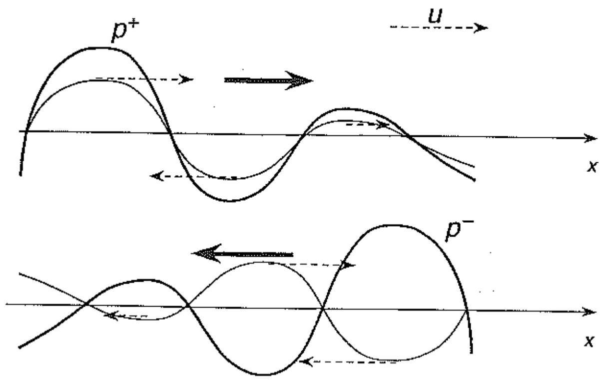


**Fig. 3.10** Illustration of the relation between pressure and particle velocity in progressive plane waves travelling in opposite directions.

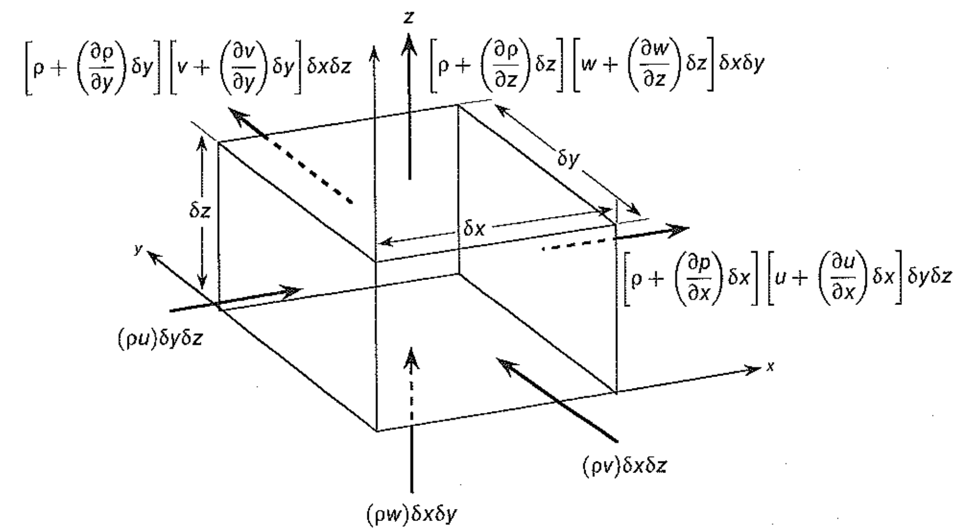


**Fig. 3.11** Mass flux through a control volume.

volume described in rectangular Cartesian coordinates $x$, $y$, $z$, together with the mass fluxes through the six faces associated with the three associated components of the velocity vector $u$, $v$, $w$. In analogue to Eq. (3.8), conservation of mass is satisfied by the equation

$$\begin{aligned}(\partial\rho/\partial t)\,\delta x\delta y\delta z &= \left[\rho u - \left(\rho + \frac{\partial \rho}{\partial x}\,\delta x\right)\left(u + \frac{\partial u}{\partial x}\,\delta x\right)\right]\,\delta y\delta z \\ &+ \left[\rho v - \left(\rho + \frac{\partial \rho}{\partial y}\,\delta y\right)\left(v + \frac{\partial v}{\partial y}\,\delta y\right)\right]\,\delta x\delta z \\ &+ \left[\rho w - \left(\rho + \frac{\partial \rho}{\partial z}\,\delta z\right)\left(w + \frac{\partial w}{\partial z}\,\delta z\right)\right]\,\delta x\delta y\end{aligned} \tag{3.32}$$

of which the linearized form is

$$\partial\rho'/\partial t = -\rho_0\,(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) \tag{3.33}$$

The term $\partial u/\partial z + \partial v/\partial y + \partial w/\partial z$ is termed the *divergence* of the velocity vector. In vector notation, $\nabla$ 'del' is a vector operator expressed as $(\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$ in which **i**, **j**, **k** are the unit vectors in the three coordinate directions. The scalar product of $\nabla$ and the velocity vector $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$, expressed as $\nabla.\mathbf{u}$, yields the divergence, which is a scalar quantity.

Figure 3.12 shows the pressures acting on the faces of the element. Linearization of the equations expressing N2LM in the three coordinate directions, performed in the same manner as for the one-dimensional case, yields

$$\partial p/\partial x = -\rho_0\,\partial u/\partial t \tag{3.34a}$$

$$\partial p/\partial y = -\rho_0\,\partial v/\partial t \tag{3.34b}$$

$$\partial p/\partial z = -\rho_0\,\partial w/\partial t \tag{3.34c}$$



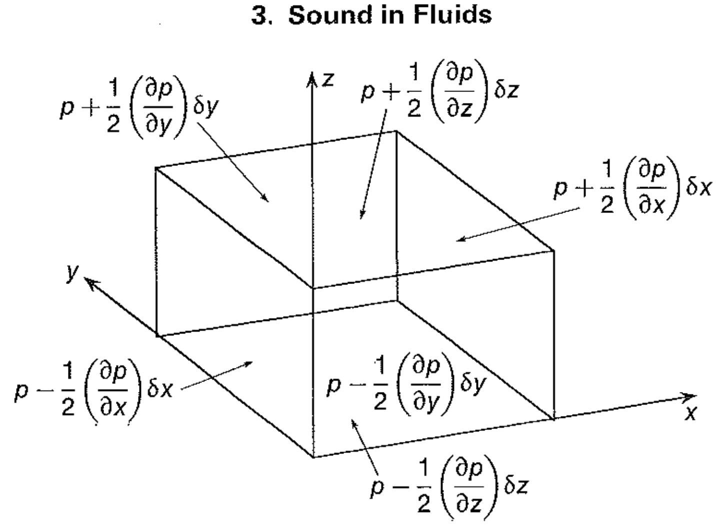


**Fig. 3.12** Fluid element subjected to pressure gradients.

These equations may be expressed in more compact form by

$$\nabla p = -\rho_0\,\partial\mathbf{u}/\partial t \tag{3.34d}$$

which is termed the 'gradient' of $p$.

Differentiation of Eq. (3.33) with respect to time, and of Eqs (3.34a), (3.34b) and (3.34c) with respect to $x$, $y$ and $z$, respectively, yields the linearized wave equation for sound pressure expressed in terms of rectangular Cartesian coordinates:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{3.35}$$

The left-hand side may be abbreviated by the use of the Laplacian scalar operator $\nabla\cdot\nabla = \nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, so that

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{3.36}$$

Students of fluid dynamics may recall that, for *incompressible* fluids, in which the speed of sound is infinite, Laplace's equation is written $\nabla^2 p = 0$.

### 3.9.6 Plane waves in three dimensions

Plane sound waves are not one-dimensional waves since they exist in three dimensions. They are functions of a single space variable if we choose that coordinate to coincide with the direction of propagation. However, we shall wish to analyse the behaviour of plane waves in systems in which this convenient choice is inappropriate. Hence we need to introduce a formalism for representing plane wave propagation in some arbitrary direction in three-dimensional space. Note that, for this special form of wave, we do not need to use Eq. (3.36) because the *selected coordinate system does not change the physics of wave propagation.* We may use the general solution to the plane wave equation (Eq. (3.19)) and simply transform it into an expression in terms of the three rectangular

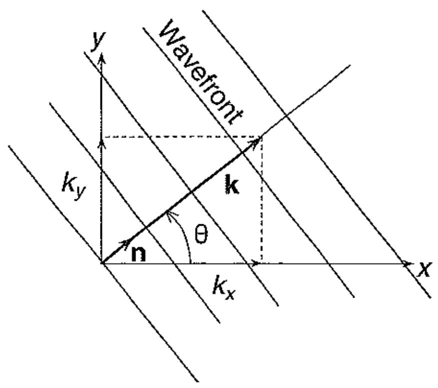


**Fig. 3.13** A plane wavefront in two dimensions.

Cartesian coordinates. The pressure in a plane wave propagating in the direction of a general linear coordinate $s$ can be expressed as

$$p(s, t) = f(ct - s) \tag{3.37}$$

which may also be expressed as

$$p(\mathbf{r}, t) = f(ct - \mathbf{r.n}) \tag{3.38}$$

in which the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and $\mathbf{n} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}$ is the unit vector defining the direction of propagation.

Under some conditions, for example in bubbly gas–liquid mixtures, sound waves are dispersive (that is to say that the phase speed is frequency dependent), and then it is more convenient to consider harmonic waves. For this purpose we define a wavenumber vector $\mathbf{k} = k\mathbf{n}$ where the magnitude $k$ equals $\omega/c$ and the direction $\mathbf{n}$ is normal to the planes of uniform phase (the harmonic wavefronts). Since the wavenumber vector may be decomposed into Cartesian components as $\mathbf{k} = k\cos\theta\,\mathbf{i} + k\sin\theta\,\mathbf{j}$, the explicit expression for a wavefront in terms of the $x$, $y$ coordinate system is $\mathbf{k.r} = (k\cos\theta)\ x + (k\sin\theta)\ y =$ constant as illustrated by Fig. 3.13. The general form of expression of spatial phase $\mathbf{k.r}$ applies to *plane* waves in any dimension and coordinate system, and a condition of constancy of this product defines a wavefront surface in any harmonic field. It should be carefully noted that *wavelengths* should never be similarly decomposed into components because, in addition to formal incorrectness, attempts so to do can easily lead to errors of interpretation. Note also that the wave itself is not decomposed into components that can be summed to restore the whole; the wavenumber vector components appear in exponential terms that are *multiplied* together to form the complete expression.

Interference fields in two- and three-dimensional space exhibit complex spatial distributions of particle velocity and, as we shall see in Chapter 5, of energy flow. The following expression represents a pressure field formed by the superimposition of four co-harmonic plane waves having wavenumber vectors parallel to the $x$–$y$ plane and $\theta = \pi/4$, as shown in Fig. 3.14:

$$\begin{aligned} p(x, y, t) = {} & \tilde{A}\exp[j(\omega t - (k\cos\theta)\,x - (k\sin\theta)y)] \\ & + \tilde{A}\exp[j(\omega t - (k\cos\theta)\,x + (k\sin\theta)y)] \\ & + \tilde{A}\exp[j(\omega t + (k\cos\theta)\,x - (k\sin\theta)y)] \\ & + \tilde{A}\exp[j(\omega t + (k\cos\theta)\,x + (k\sin\theta)y)] \end{aligned} \tag{3.39}$$



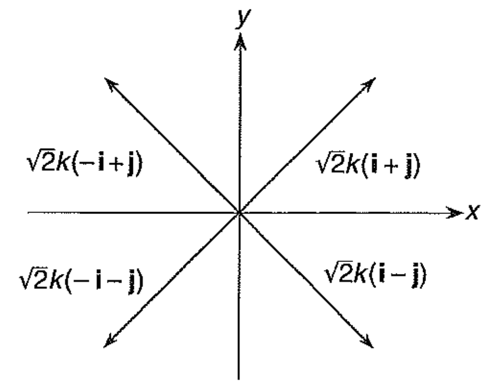


**Fig. 3.14** Wavenumber vectors of four plane waves.

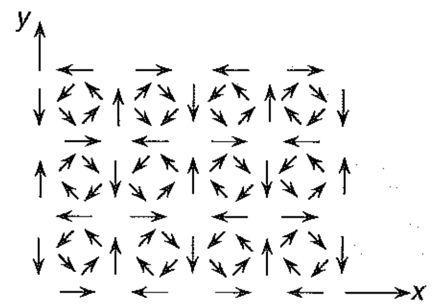


**Fig. 3.15** Particle velocity field in the interference field produced by the four waves.

An expression for the particle velocity field can be obtained by applying Eqs (3.34a) and (3.34b), together with the relation between particle acceleration and velocity in a harmonic field. The vector field is shown in Fig. 3.15.

Plane waves are particularly simple in form, but they are very important. This is not only because they predominate in many cases of practical interest to the acoustical engineer, for example in tubes, pipes and ducts, but also because *any* form of sound field may be expressed as an infinite sum of plane waves, albeit that some may have imaginary wavenumbers. This fact is central to the measurement technique known as nearfield acoustic holography (NAH) by means of which sources of sound such as diesel engine structures can be imaged by making measurements of sound pressure at an array of points distributed over a plane at a short distance from the source [3.2]. NAH is widely used by automotive manufacturers to detect sources of unacceptably high levels of noise generated by their products. However, this synthetic form of representation is not helpful for studying and understanding many fundamental problems in acoustics, and we now turn to another equally important non-plane wave solution to the wave equation.

### 3.9.7 The wave equation in spherical coordinates

Coordinate systems other than the rectangular Cartesian system may be used to derive alternative forms of the wave equation. Although this could be done from first principles

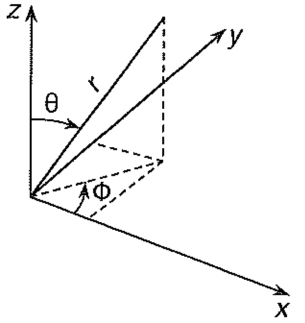


**Fig. 3.16** Spherical coordinates.

by deriving alternative forms of the equations of conservation of mass and momentum appropriate to each coordinate system, it is simpler to apply standard coordinate transformation formulae, as found in mathematical textbooks, to Eq. (3.36). The form most appropriate to any problem is selected on the basis of the geometrical form of any sources or fluid boundaries present. Since no one direction in an unbounded uniform isotropic fluid is any different from any other, a sound wave generated by a very localized disturbance of fluid density spreads out in all directions; waves produced on the surface of a pond by the entry of a small pebble provide a two-dimensional analogue. Such waves can be represented by infinite sums of plane waves, but it makes much more sense to select, respectively, spherical and cylindrical polar coordinates in which to express these wavefields.

Figure 3.16 shows a spherical coordinate system. The Laplacian in Eq. (3.36) becomes

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)\frac{\partial^2}{\partial \phi^2}$$

and the components of the gradient operator become

$$\nabla_r = \frac{\partial}{\partial r}, \qquad \nabla_\theta = \frac{1}{r}\frac{\partial}{\partial \theta} \qquad \text{and} \qquad \nabla_\phi = \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}$$

### 3.9.8 The spherically symmetric sound field

In Chapter 6 we shall discover that any form of sound source may be represented by an array of elementary sources that, in isolation, radiate uniformly in space. It therefore makes sense to transform Eq. (3.36) into spherical coordinates and to assume that the acoustic variables are functions only of time and the single radial coordinate.

The Laplacian reduces to $(1/r^2)\,\partial/\partial r\,(r^2\partial/\partial r)$ and Eq. (3.36) becomes

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\left(\frac{\partial p}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{3.40}$$

This equation has the same form as the plane wave equation (3.17) with $p$ replaced by $rp$. Hence, the general solution follows from Eq. (3.18):

$$p(r, t) = \frac{1}{r}[f(ct - r) + g(ct + r)] \tag{3.41}$$

The function $f(ct - r)$ represents a wave travelling outwards from the coordinate origin



and $g(ct + r)$ represents a wave converging on the origin. The latter is rare, but is of importance in lithotripsy, in which ultrasound is focused on a kidney by a ring of radiators in order to fragment stones within. The converging wave will not be considered further. Plane and spherically propagating wavefronts travel at the same frequency-independent speed. Therefore the sound pressure–time history takes the same *form* at any point in the fields; but, in the case of spherical propagation, the magnitude of the outward-going acoustic disturbance *decreases linearly with distance* from the origin. The corresponding sound pressure level decreases by 6 dB per doubling of distance. Chapter 5 shows that, in both these forms of field, the rate of transport of acoustic energy per unit area of wavefront (known as the sound intensity) is proportional to the square of the sound pressure. The product of the area of a spherical surface (proportional to $r^2$) and the sound intensity (proportional to $r^{-2}$) is independent of $r$, thereby satisfying the requirement for conservation of energy. The solution given by Eq. (3.41) is not valid at $r = 0$. This problem does arise in the application of a general solution to the problem of sound radiation from vibrating bodies that will be introduced in Chapter 6. However, alternative forms of solution are available for overcoming this problem, some of which are exploited in commercial computer software.

### 3.9.9 Particle velocity in the spherically symmetric sound field

Transformation of the equation of conservation of momentum from rectangular to spherical coordinates yields a linearized relation between radial particle acceleration and radial pressure gradient of the same form as for the plane wave:

$$\partial p/\partial r = -\rho_0\,\partial u_r/\partial t \tag{3.42}$$

There is no tangential component of fluid motion in a spherically symmetric field.

Explicit solution of Eqs (3.40) and (3.42) requires specific forms of the function $f$, of which the analytically most illuminating and practically most useful (thanks to Fourier) is the time-harmonic form

$$p(r, t) = \frac{\tilde{A}}{r}\exp[j(\omega t - kr)] \tag{3.43}$$

The associated radial particle velocity is given by Eq. (3.42) as

$$u_r = \frac{1}{j\omega}\frac{\partial u_r}{\partial t} = \frac{j}{\omega\rho_0}\frac{\partial p}{\partial r} \tag{3.44}$$

The relation between complex amplitudes of pressure and particle velocity is

$$\tilde{p}/\tilde{u}_r = \rho_0 c[jkr/(1 + jkr)] \tag{3.45}$$

which is illustrated by Fig. 3.17. Now we see a crucial difference between the plane travelling wave field and the outgoing spherical waves field. In the former, the particle velocity is linearly proportional to, and in phase with, the pressure. The latter exhibits the same relation at positions at a great distance from the origin compared with a wavelength, where $kr \gg 1$. But at distances where $kr \ll 1$, the magnitude of the ratio of pressure to particle velocity is much less than the plane wave value of $\rho_0 c$ and the relative phase approaches $\pi/2$ as $kr$ tends to zero. In relation to sound fields generated by the elementary model of an omnidirectional source, treated in detail in Chapter 6, and known as a 'monopole', this characteristic leads to the concepts of a 'near field' and a 'far

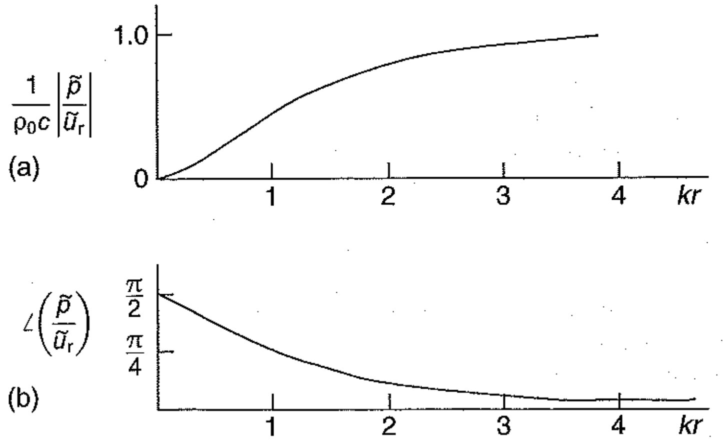


**Fig. 3.17** Relation between pressure and radial particle velocity in a spherically symmetric field: (a) magnitude; (b) phase.

field'. Consideration of the energetic aspects of such a field in Chapter 5 yields the concepts of the 'reactive' and 'active' components of the field. This terminology refers to the fact that sound energy is transported by the action of sound pressure on moving particles. In a harmonic, spherically symmetric, sound field, energy is transported to the far field only by the cooperation of the sound pressure and that component of particle velocity that is *in phase* with the pressure. Cooperation between the pressure and the quadrature component of particle velocity produces a localized form of energy transport in which the associated local energy density *oscillates* between purely kinetic and potential states, without net transport. This latter phenomenon is characteristic of the reactive component of the field.

### 3.9.10 Other forms of sound field

Sound sources and acoustic environments are infinitely diverse; so are the forms of sound field that they produce. However, the two particular forms of field treated in this chapter are generic in as much as they can be harnessed to form the building blocks of a wide range of fields of practical interest. One general form that has not been dealt with is the cylindrically spreading field such as that which would be generated by a uniformly pulsating, infinitely long, circular section tube. It is a curiosity of wave fields that such a field exhibits a more complicated form of space–time dependence than plane or spherical fields. This is because, at any one observation point in the radiation field of a tube that undergoes a very brief (impulsive) expansion or contraction, the initial disturbance that arrives from the nearest part of the source is followed by progressively smaller disturbances that arrive sequentially from more and more remote points on the tube: the field is therefore said to exhibit a 'tail'. In the special case of harmonic pulsation this phenomenon creates a radial dependence of the acoustic variables. This dependence is mathematically described by Hankel functions, which are a particular solution to Bessel's differential equation. Although presentation of a detailed analysis of this form of field is not considered to be appropriate in this book, it is of considerable practical importance in the mathematical modelling and analysis of sound propagation and



radiation from vibrating pipes, which constitute a significant proportion of serious industrial noise sources. Treatment of radiation from tubes can be found in *Sound, Structures and their Interaction* (Junger and Feit, 1986), cited in the Bibliography. It is also relevant to the propagation and control of sound generated by linearly extended sources such as dense road traffic. In relation to the traffic noise problem, it is important to know that the rate of reduction with radial distance of the sound pressure level generated by a line source is, in theory, 3 dB per doubling of distance, but is nearer 4 dB in practice. Noise barriers are also less effective in countering line source noises than point source noises.

## Questions

3.1 By perturbing each variable in the Ideal Gas Equation (3.1) and in the equation expressing the adiabatic relation between pressure and density, and neglecting second-order quantities, show that the relation between fractional change of temperature and fractional change of pressure is $\delta T/T_0 = (p/P_0)\,[(1 - 1/\gamma]$.

3.2 The complex amplitudes of pressure in oppositely-directed plane waves are $\tilde{A} = 1 + 3j$ and $\tilde{B} = 2 - 2j$. Derive expressions for the real pressure amplitude and mean square pressure as function of position in the interference field.

3.3 What is the ratio of maximum to minimum mean square pressure in the field specified in Question 3.2? Express it as a ratio and in terms of dB.

3.4 A large number of polystyrene foam balls of 3 mm diameter is distributed randomly throughout a volume of water. Assuming that they have the same bulk modulus as air, and that the average density is $10^5$ balls per $m^3$, estimate the approximate speed of sound in the compound medium.

3.5 The sound pressure in a harmonic sound field is expressed as $p = \tilde{A}\exp[j(\omega t - kx)]$, where $\tilde{A} = (0.5 + 0.5j)$. Derive an expression for $p^2(x, t)$.

3.6 Evaluate the root mean square (rms) pressure and density, together with particle displacement, velocity and acceleration, in a 250 Hz plane travelling wave of which the sound pressure level is 74 dB (see Appendix 6).