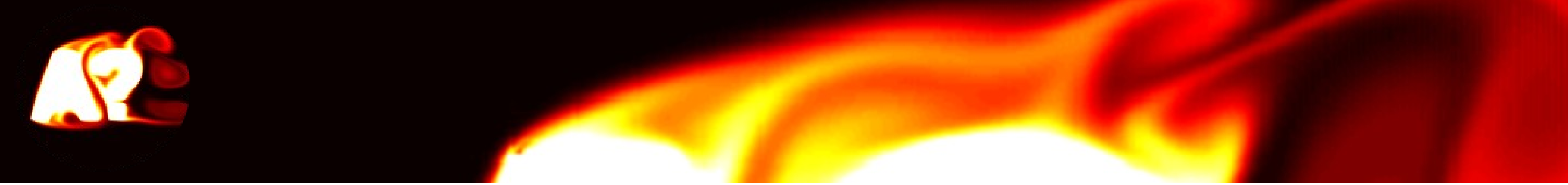


EEN-1020 Heat transfer

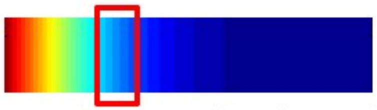
Week 2: Fins, 2d Conduction, Thermal Resistance, and Numerical Solution in 2d

Prof. Ville Vuorinen

October 31st- Nov. 1st 2023

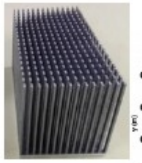
Aalto University, School of Engineering

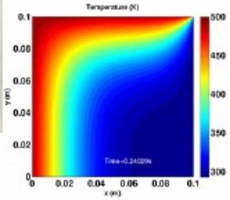
Week 1: Energy conservation, heat equation, conduction Fourier/Newton



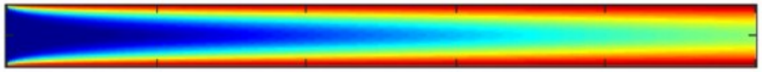
$$\frac{\partial T}{\partial t} \approx \frac{1}{\Delta x}\left(k\frac{\partial T(x+\Delta x/2,t)}{\partial x} - k\frac{\partial T(x-\Delta x/2,t)}{\partial x}\right)$$

Week 2: Fin theory, conduction, intro to convection



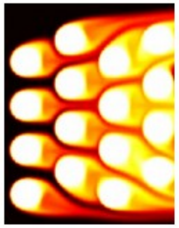


Week 3: convective heat transfer – internal flow (channel)



Week 4: convective heat transfer – external flow (fin systems)

Week 5: natural convection, boiling, correlations



On the heat transfer course, we have "5 friends" i.e. 5 main principles that are used to explain heat transfer phenomena

1) Energy conservation: "J/s thinking"
2) Fourier's law
3) Newton's cooling law
4) Energy transport equation – convection/diffusion equation
5) Momentum transport equation – Navier-Stokes equation

Lecture 2.1 Theory: Fins and thermal resistance

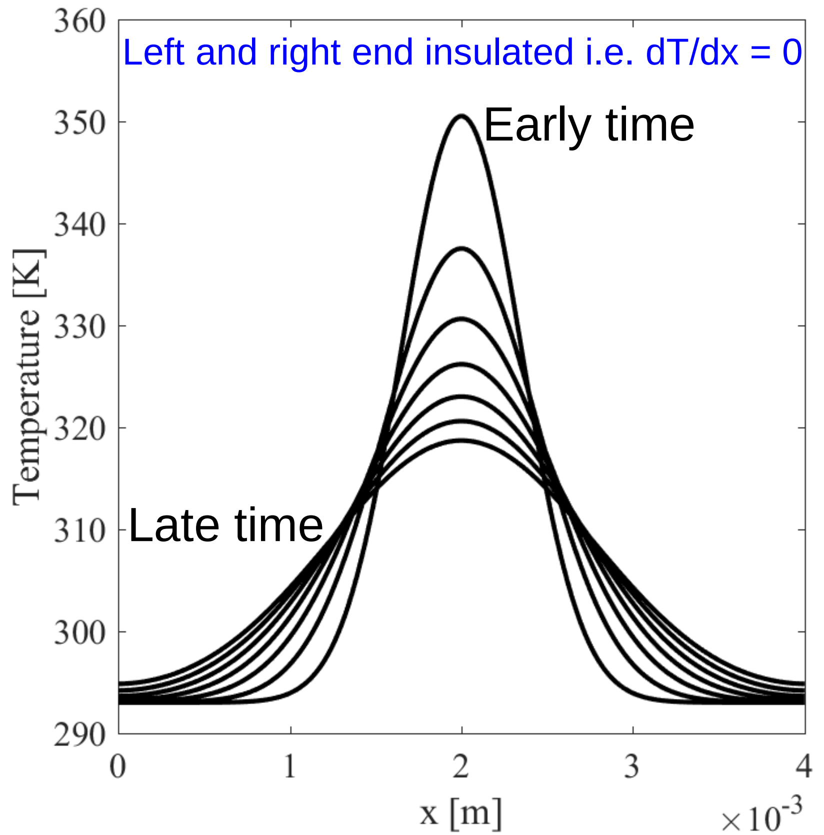
ILO 2: Student can apply Fourier's law and Newton's law in fin theory and thermal resistance context. Further, the student can analyse 2d heat transfer data in Matlab and formulate an energy balance for 2d system.

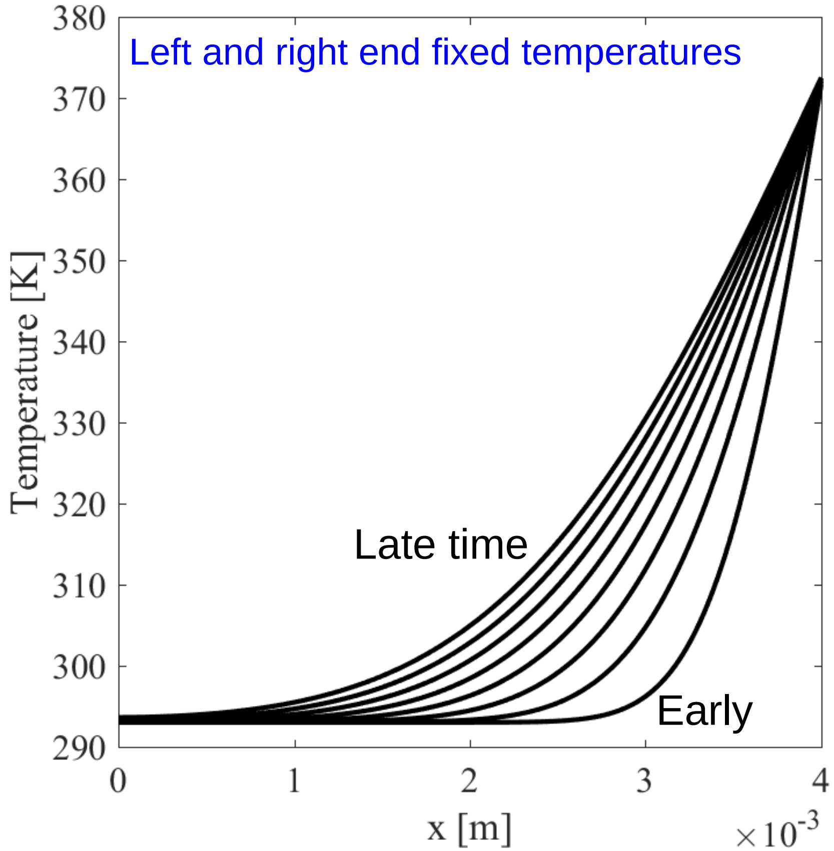
Thermal resistance

Two examples of heat diffusion in 1d

- **Left:** initially Gaussian temperature profile diffuses. Amplitude decreases and the distribution spreads with time. The domain ends are insulated → heat does not escape from the domain. $q_{left} = q_{right} = 0$. **Note:** for fixed q bc T results.

- **Right:** initially constant temperature object is heated from right end. Temperature diffuses to the left end. Both ends are at fixed temperatures. $T_{left} = 293K$ and $T_{right} = 373K$. **Note:** for fixed T bc q results.





Two examples of heat diffusion in 1d

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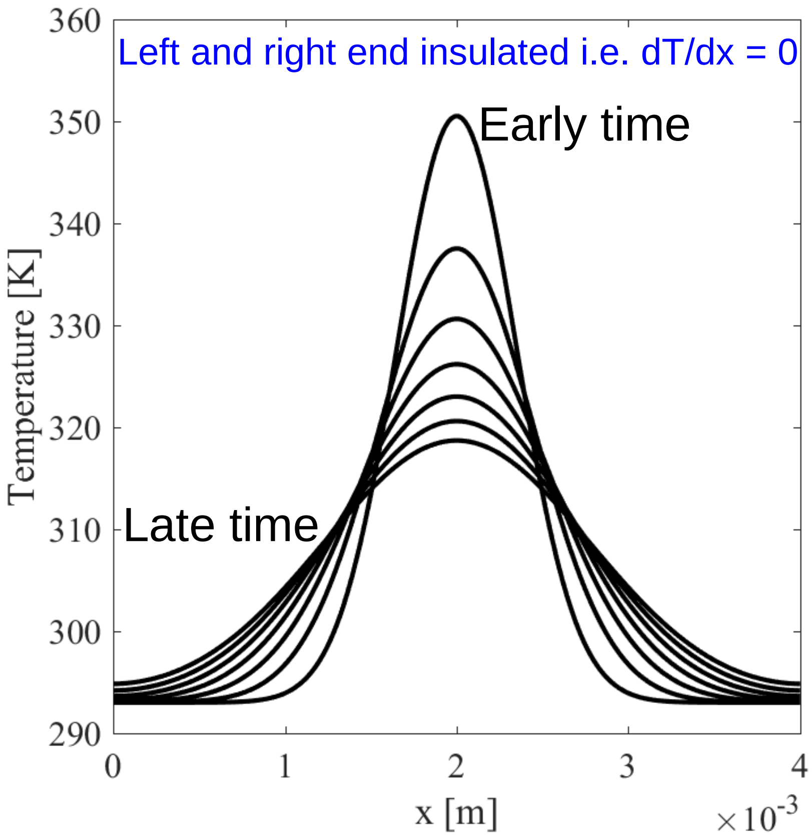
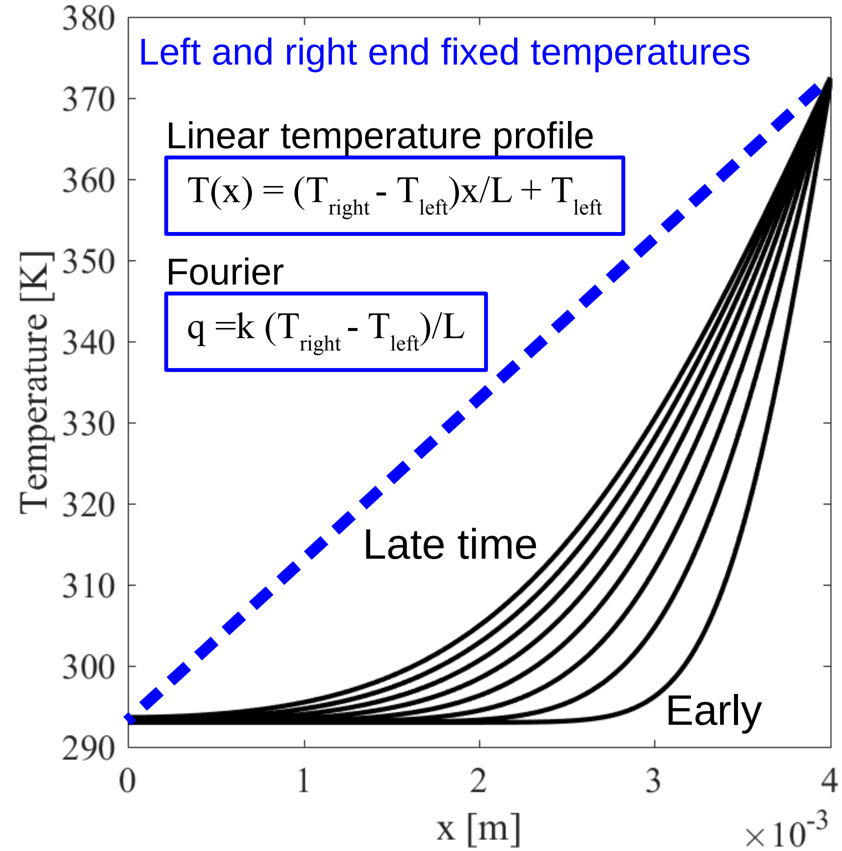




Recall from last week: example → "energy efficient window"

More difficult: know surface temperatures in practice

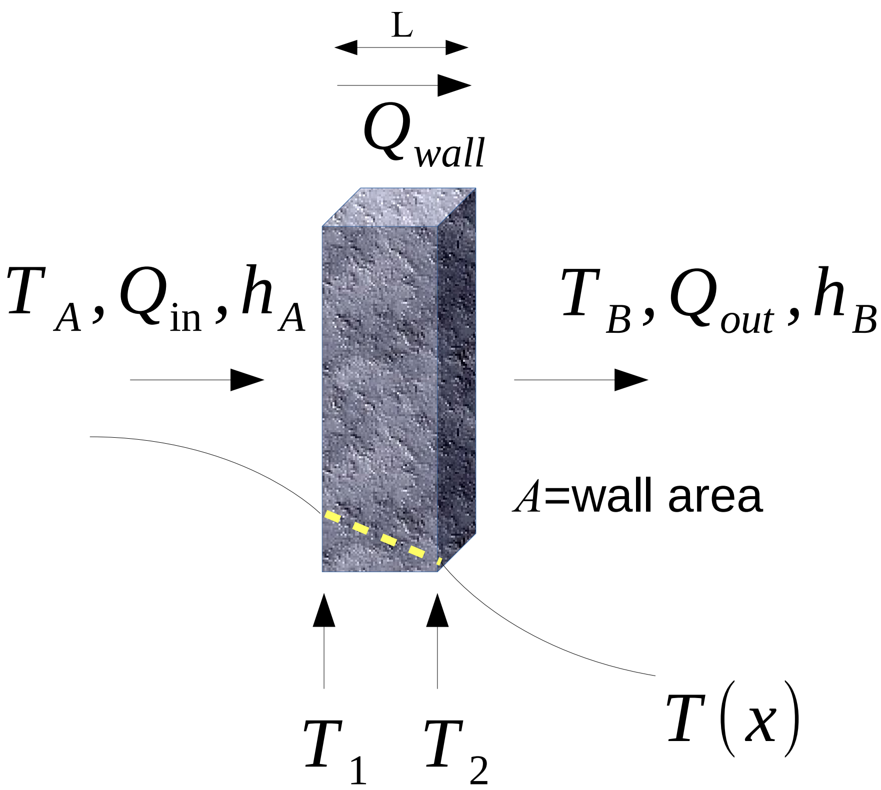
Remark 1: if we know surface temperatures T_1 and T_2 on both sides of a window it is easy to calculate escaping heat flux using Fourier's law But, in practice we seldom know those surface temperature values.

Remark 2: T_1 and T_2 do not correspond to the indoor room temperature and outdoor temperature values. Also airflow on boths sides of the windows will affect the actual surface temperature values.

Remark 3: It is easy to measure T_{room} and T_{out}

More easy: know "bulk" temperatures of working fluids.

Derivation of steady state heat rate ([Q]=W=J/s) through a wall. Convective heat transfer coeff. h (wind&indoor ventilation)



In steady state:

$$Q_{in} = Q_{wall} = Q_{out} = Q$$

Newton's law:

$$Q/(h_A A) = (T_A - T_1) \quad (1)$$

Fourier's law:

$$Q/(kA/L) = (T_1 - T_2) \quad (2)$$

Newton's law:

$$Q/(h_B A) = (T_2 - T_B) \quad (3)$$

Sum: (1) + (3) and note that T_2-T_1 appears i.e. - (2)

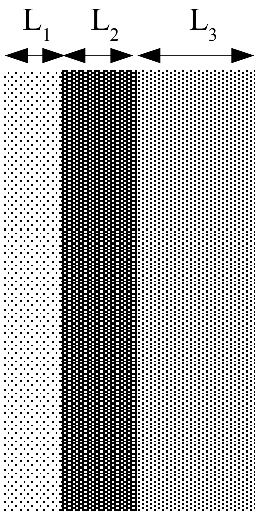
$$T_A - T_B + T_2 - T_1 = Q/(h_A A) + Q/(h_B A)$$

Wall heat rate (W):

$$Q = \frac{T_A - T_B}{1/(kA/L) + 1/(h_A A) + 1/(h_B A)}$$

Note: If you know Q you also know T_1 and T_2 and T(x).

Thermal resistance – composite wall with multiple (i=1,2,…,N) material layers



Heat rate in analogy with Newton's cooling law:

$$Q = UA\Delta T\text{, with } \Delta T = T_A - T_B$$

$$Q = \frac{A\Delta T}{\Sigma_i 1/(k_i/L_i) + 1/(h_A) + 1/(h_B)}$$

Overall heat transfer coefficient ([U]=W/m²K):

$$U = \frac{1}{\Sigma_i 1/(k_i/L_i) + 1/(h_A) + 1/(h_B)}$$

Thermal resistance ([R]=K/W):

$$R_{tot} = \frac{1}{UA}$$

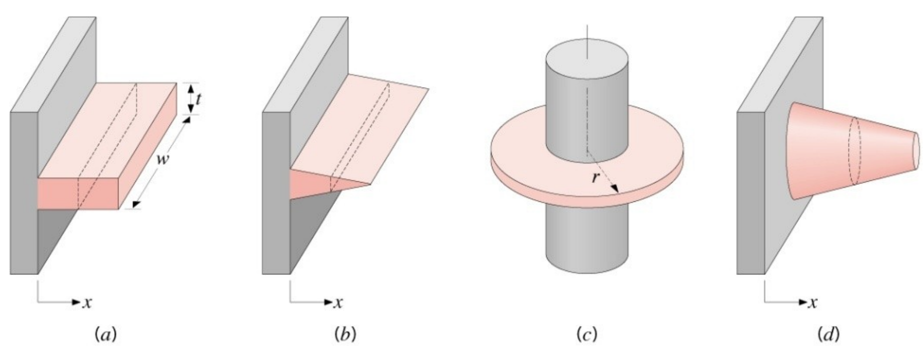
Some benefits of thermal resistance concept:

→ design of thermal insulation (buildings, clothes, combustion)
→ allows to maximize or minimize heat flux
→ allows designs to avoid hot pools of temperature from forming
→ allows to design temperature profiles (e.g. avoid condensation)

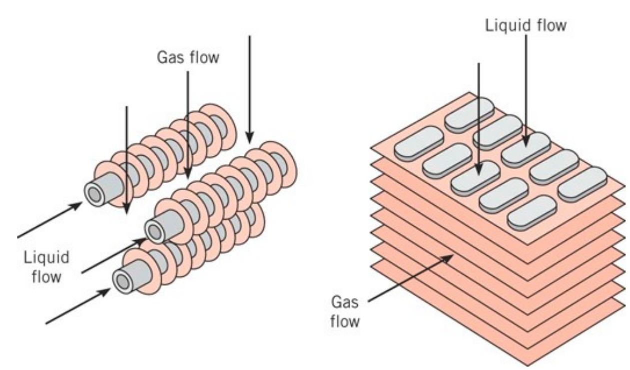
Fin theory

Fins and fin theory

- To enhance heat transfer between **solid** and **fluid** phases
- Conduction along the fin, conduction and convection outside the fin
- **Temperature distribution inside the fin in crucial role.**
- In many circumstances $T=T(x)$ i.e. 1d temperature distribution
- It enables formulation of **1d energy balance i.e. heat equation** for a fin
- Such 1d conduction assumption in fin context is called **fin theory**.



Basic fin types



Typical finned-tube heat exchangers

Fins – surface extrusions that increase area of surface to increase heat transfer

3d printed heat exchangers intended for air cooling (V.Vuorinen, K.Kukko, K.Saari)

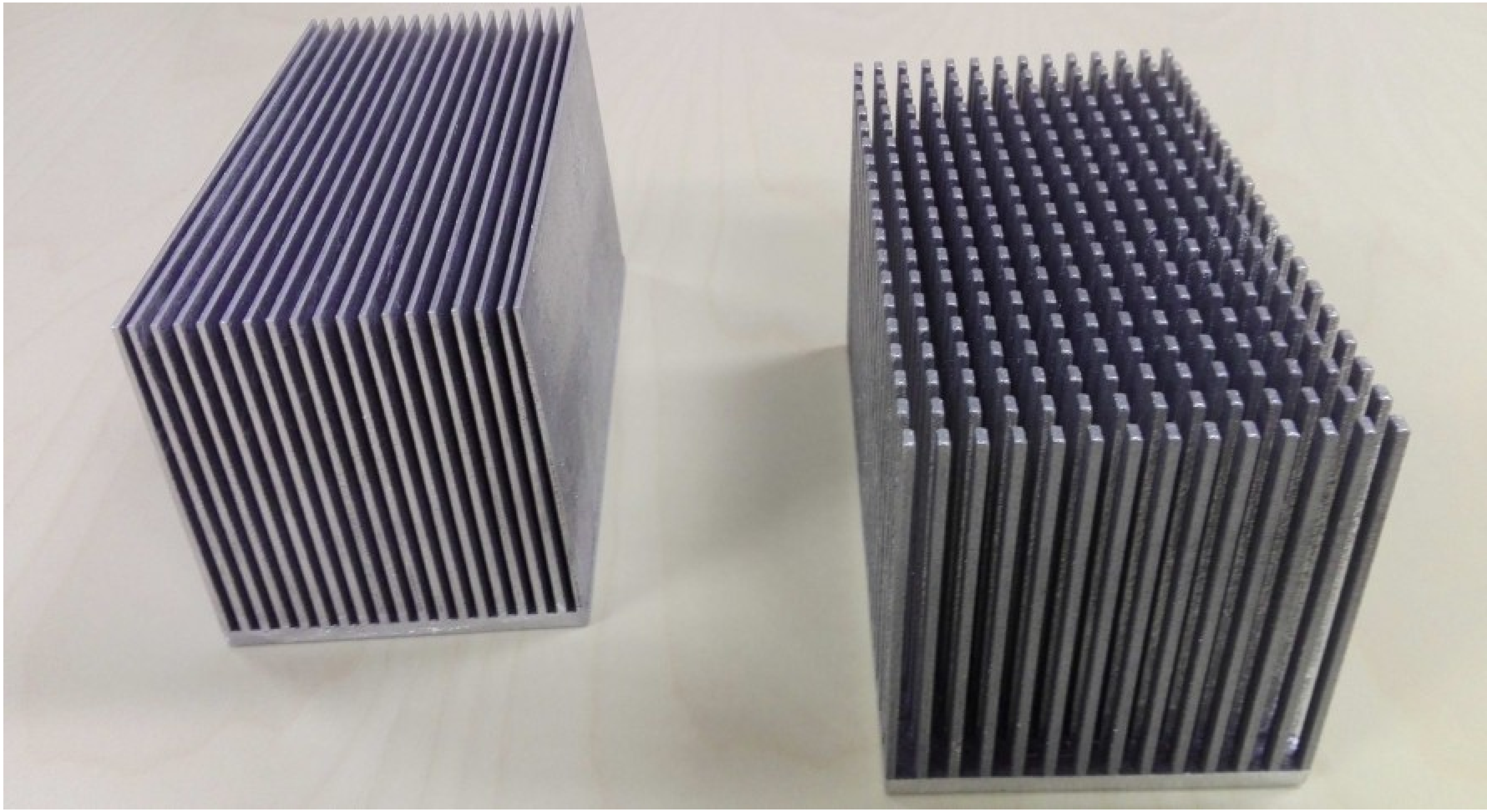


Plate fins

Pin fins

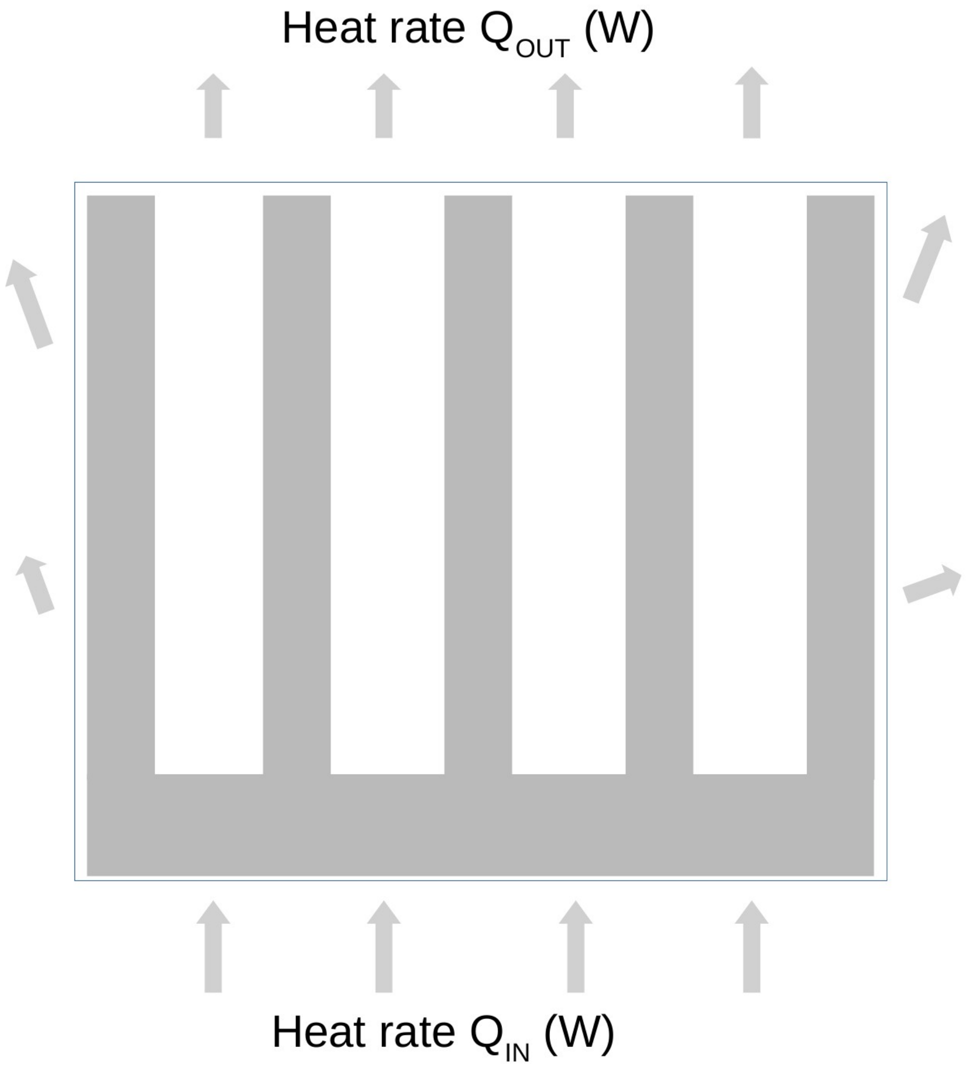
Energy balance (J/s thinking) for a heated object (mass m, specific heat c_p)

Energy balance:

$$Q_{\mathrm{IN}} - Q_{\mathrm{OUT}} = c_p\, m \frac{\Delta T_{ave}(t)}{\Delta t}$$

Steady state:

$$Q_{\mathrm{IN}} = Q_{\mathrm{OUT}}$$



Temperature distribution inside the fin in crucial role:

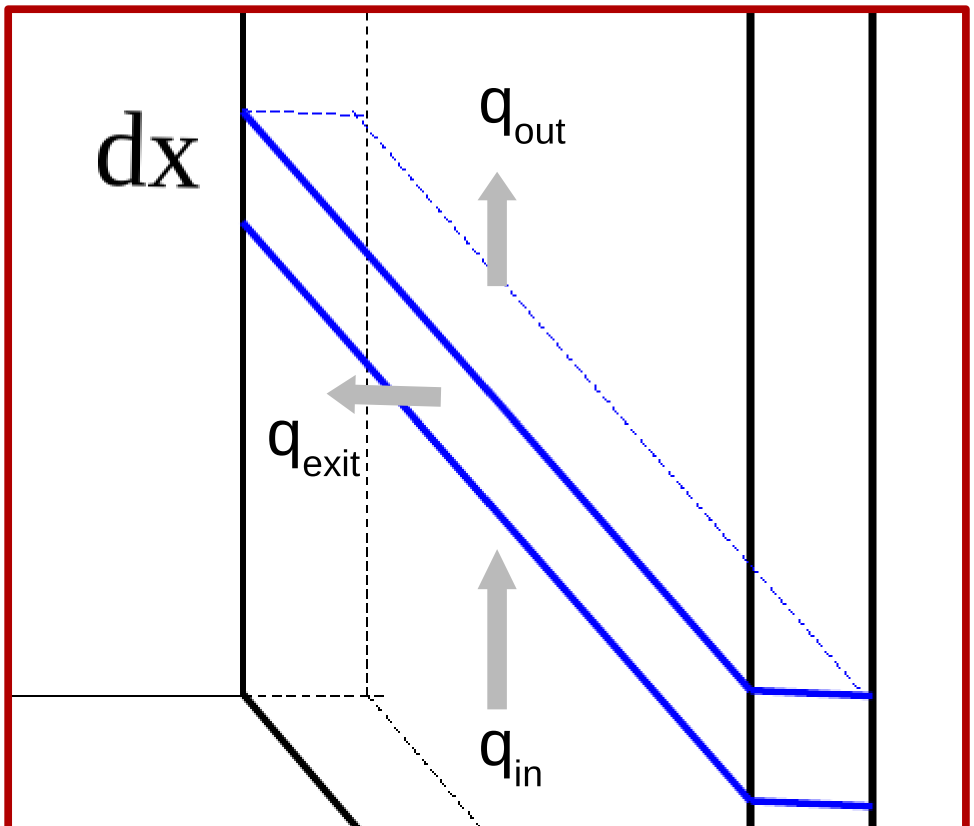
→ If we knew T=T(x) along a fin we could calculate the power which enters each fin. Also, we could try to optimize the fins & material costs to have good efficiency For heat transfer.

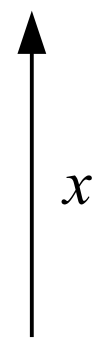
→ We would then also know the entering heat flux to the fins.

Local energy balance (J/s thinking) for a single fin

Energy balance in steady state (J/s):

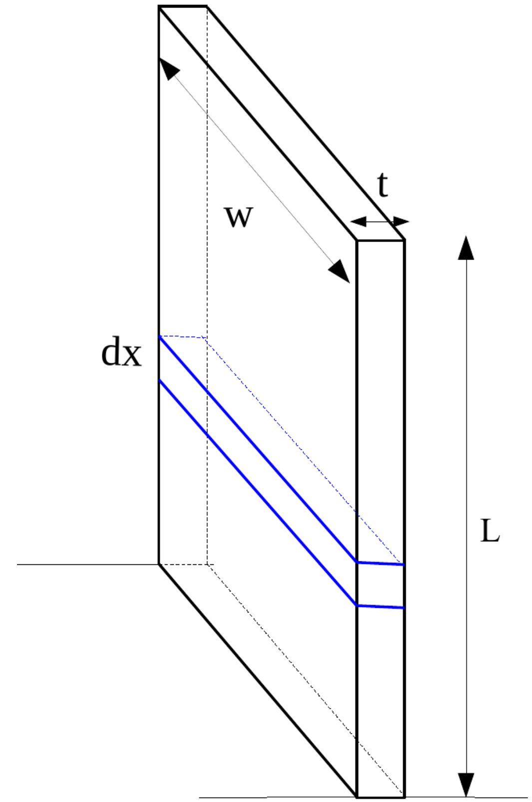
$$q_{in} - q_{out} - q_{exit} = 0$$

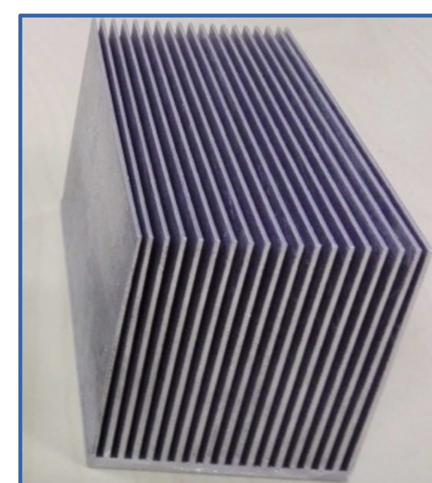




$T=T(x)$

Single fin





Local energy balance (J/s thinking) for a single fin

Fourier's law:

Energy conducts (J/s) into a small volume

$$q_{\text{in}} = -kA_c \frac{dT\left(x - dx/2\right)}{dx}$$

$$A_c = wt$$

Fourier's law:

Energy conducts (J/s) out of a small volume

$$q_{\text{out}} = -kA_c \frac{dT\left(x + dx/2\right)}{dx}$$

Newton's law:

Energy exits (J/s) from fin to fluid

(strip of area dA_s height dx, perimeter P=2(L+d))

$$q_{exit} = h dA_s \left(T - T_\infty\right)$$

$$dA_s = 2\left(w + t\right)dx$$

Local energy balance (J/s thinking) for a single fin

$$q_{\text{in}} - q_{\text{out}} - q_{\text{exit}} = 0$$

$$kA_c\left(-\frac{dT(x-dx/2)}{dx} + \frac{dT(x+dx/2)}{dx}\right) - h\,dA_s(T - T_\infty) = 0$$

$$A_c = wt$$

$$dA_s = 2(w+t)\,dx = P\,dx$$

When $\mathrm{dx} \rightarrow 0$ we get the heat equation for $\mathrm{T=T(x)}$ in the fin but now the equation has also a heat loss term as heat escapes to the fluid:

$$\frac{d^2T}{dx^2} = \frac{h\,P}{k\,A_c}(T - T_\infty)$$

$$m^2 = \frac{h\,P}{k\,A_c}$$

Definition of derivative

$$\frac{1}{\Delta x}\left(\frac{dT(x+dx/2)}{dx} - \frac{dT(x-dx/2)}{dx}\right) = \frac{d^2T(x)}{dx^2}, \text{ when } \Delta x \rightarrow 0$$

Incropera: 1d Temperature Distribution and Heat Loss Along a Fin

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q_f
A	Convection heat transfer: $h\theta(L) = -k\,d\theta/dx\|_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.70)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.72)
B	Adiabatic $d\theta/dx\|_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.75)	$M \tanh mL$ (3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.78)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx} (3.79)	M (3.80)

$\theta \equiv T - T_\infty$ $\quad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\quad M \equiv \sqrt{hPkA_c}\,\theta_b$

$$\frac{d^2T}{dx^2} = \frac{hP}{kA_c}(T - T_\infty)$$

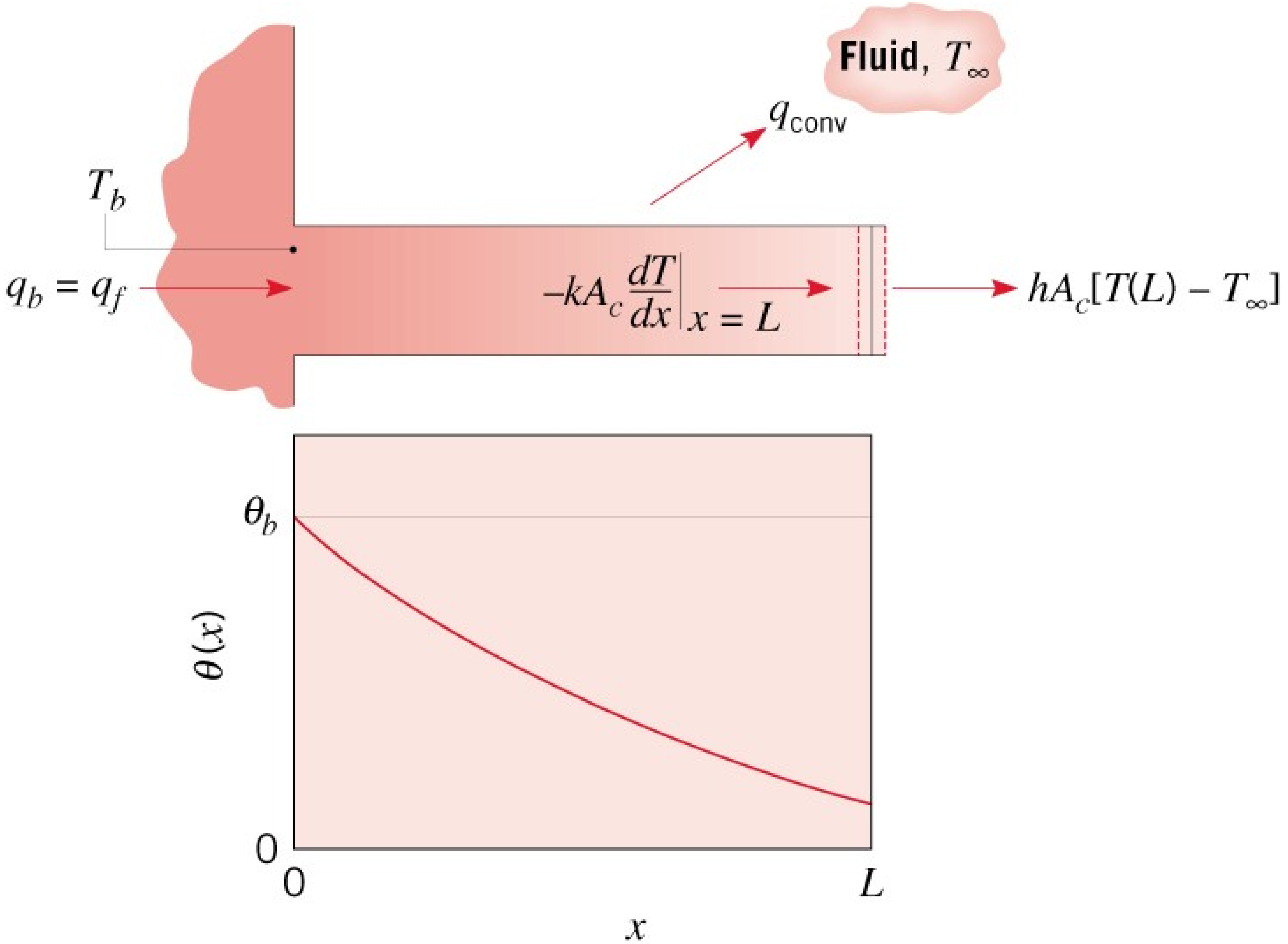


Figure 3.18 (Incropera): Conduction and convection in a fin of uniform cross section.

Example: Find temperature distribution and heat rate in a very long copper rod (diameter D=5mm) with T_b=373K and T_∞= 298K and convection coefficient due to airflow h=100W/m^{2}K

Table 3.4: For long fins →

Temperature distribution

$$T(x) \approx T_\infty + (T_b - T_\infty) e^{-mx}$$

Estimate thermal conductivity at average temperature = 335K from the Appendix (Incropera)

$$k = 398\, W/mK$$

Estimate m:

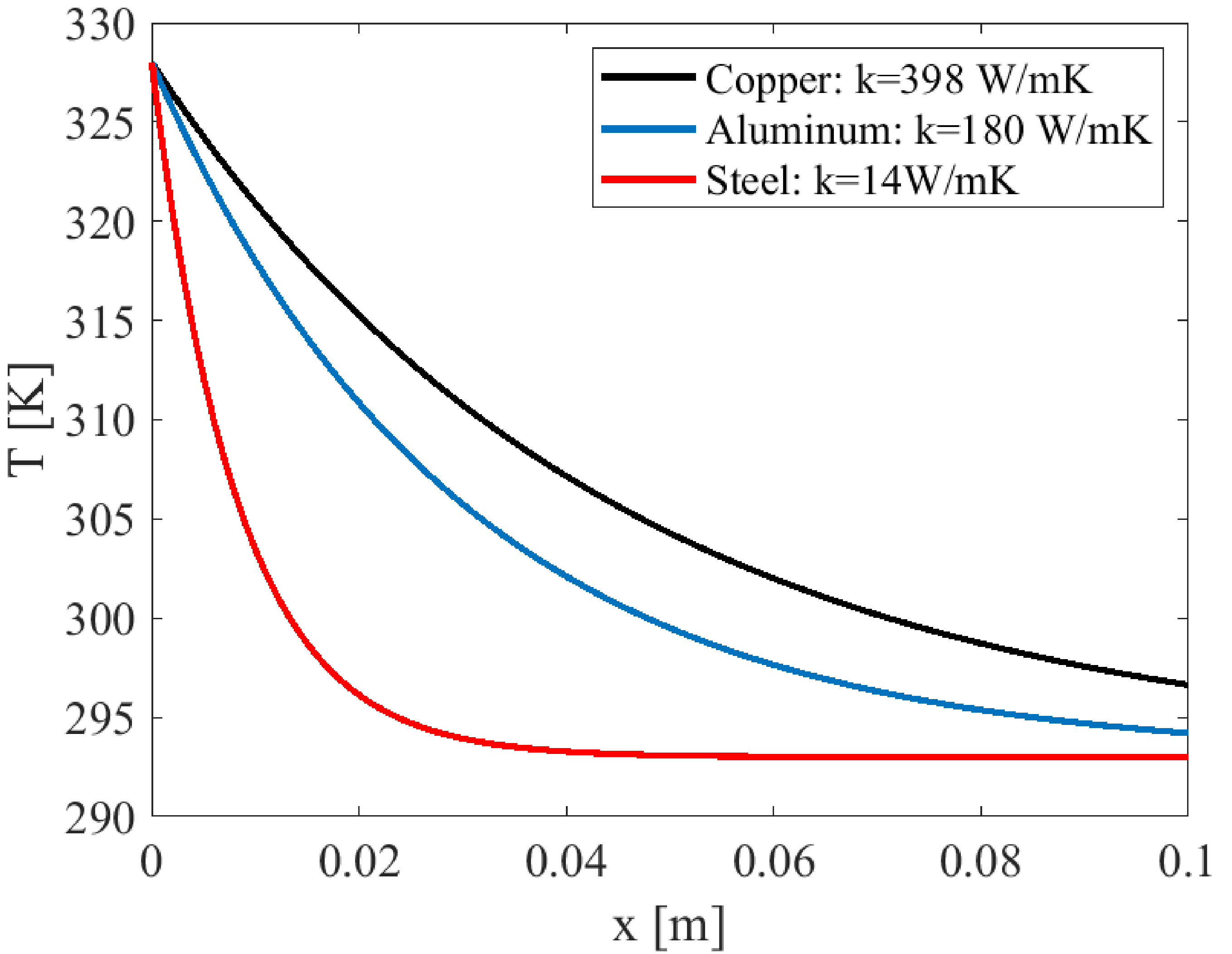
$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kD}} \approx 14.2\, m^{-1}$$

Table 3.4: Heat rate →

Heat rate

$$q_f = \sqrt{hPkA_c}\,\theta_b = 8.3\, W$$

Example temperature profiles for another infinitely long fin assuming T_b=328K, T_∞=293K



Origin of table 3.4? Write the fin heat conduction equation in more compact form and note the general solution with BC's.

$$\frac{d^2T}{dx^2}=\frac{hP}{kA_c}(T-T_\infty)$$

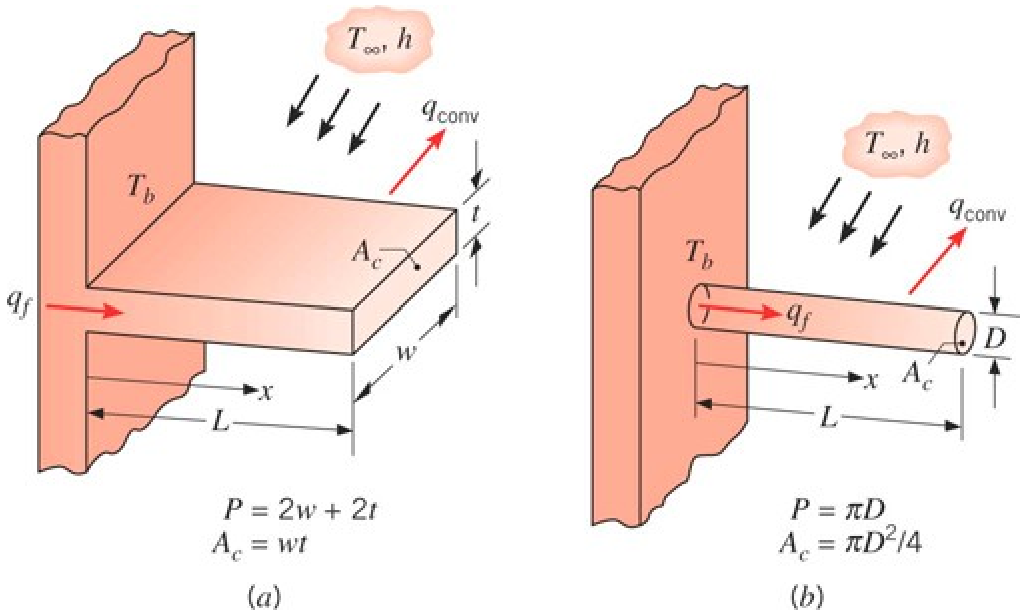
$$m^2=\frac{hP}{kA_c}$$

$$\theta=(T-T_\infty)$$

$$\frac{d^2\theta}{dx^2}-m^2\theta=0$$

→ **General solution**
(hyperbolic functions are linear comb. of exp. functions)

$$\theta=C_1e^{mx}+C_2e^{-mx}$$



→ For different boundary conditions, we can always solve the temperature distribution T(x).

→ When we know T(x), we can calculate the heat transfer rate in two different ways.

1) Fourier's law

2) Newton's cooling law

Fin effectiveness

Fin effectiveness: (heat transfer rate with fin) / (heat transfer rate without fin)

$$\epsilon_f = \frac{q_f}{hA_{c,b}\theta_b}$$

Usage of fins typically justified if > 2

Common assumption (not reality but useful)
→ assume that h is unaffected by 1) spatial position, and 2) presence of fins

Fin effectiveness (for infinitely long fin) reads (Table 3.4):

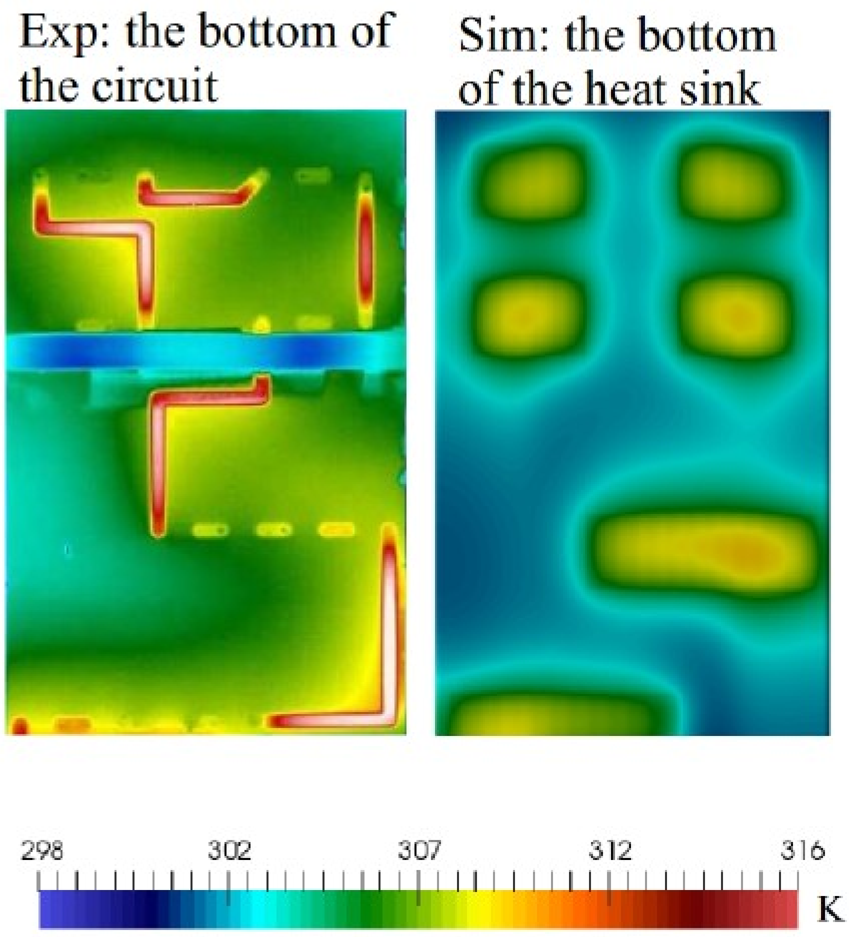
$$\epsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

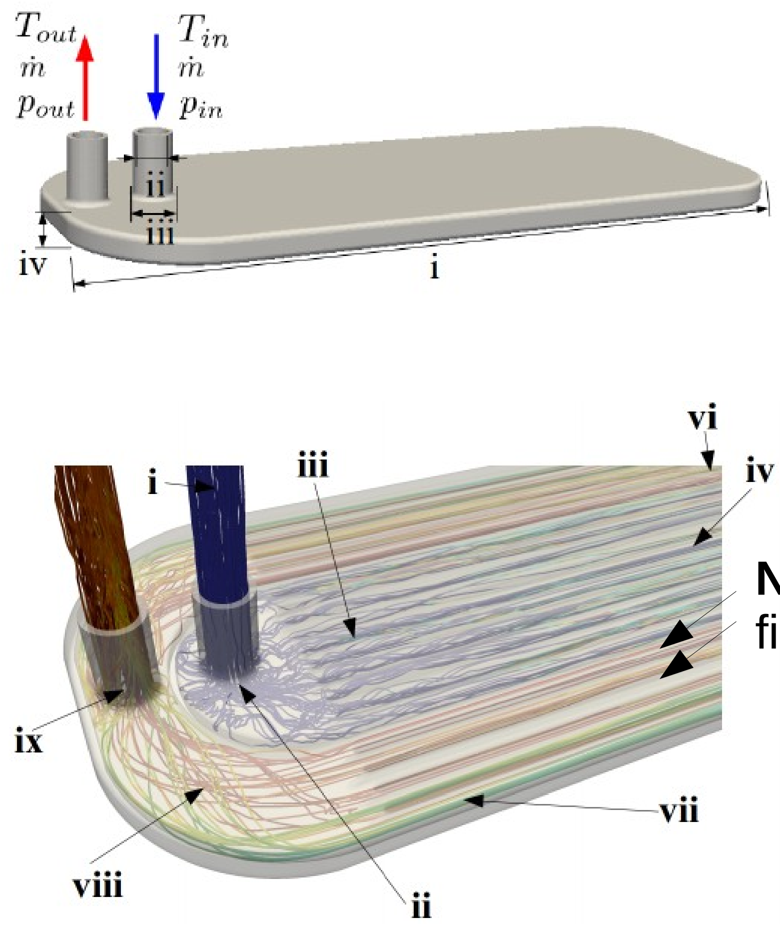
Heat transfer enhancement if:
→ perimeter to the area increased → prefer thin, closely spaced fins but not too close to not impede flow (e.g. laminarization/stagnation) between fins
→ if k/h is "small" then more need for fins (e.g. natural convection)
→ if fluid is gas then more need for fins

Example: automobile radiator (fins on the air flow side, hot water in the inside)

A few examples from our own research

Example: liquid cooling an electric circuit by placing a cooling plate with 3d printed finned microchannels on top of the circuit



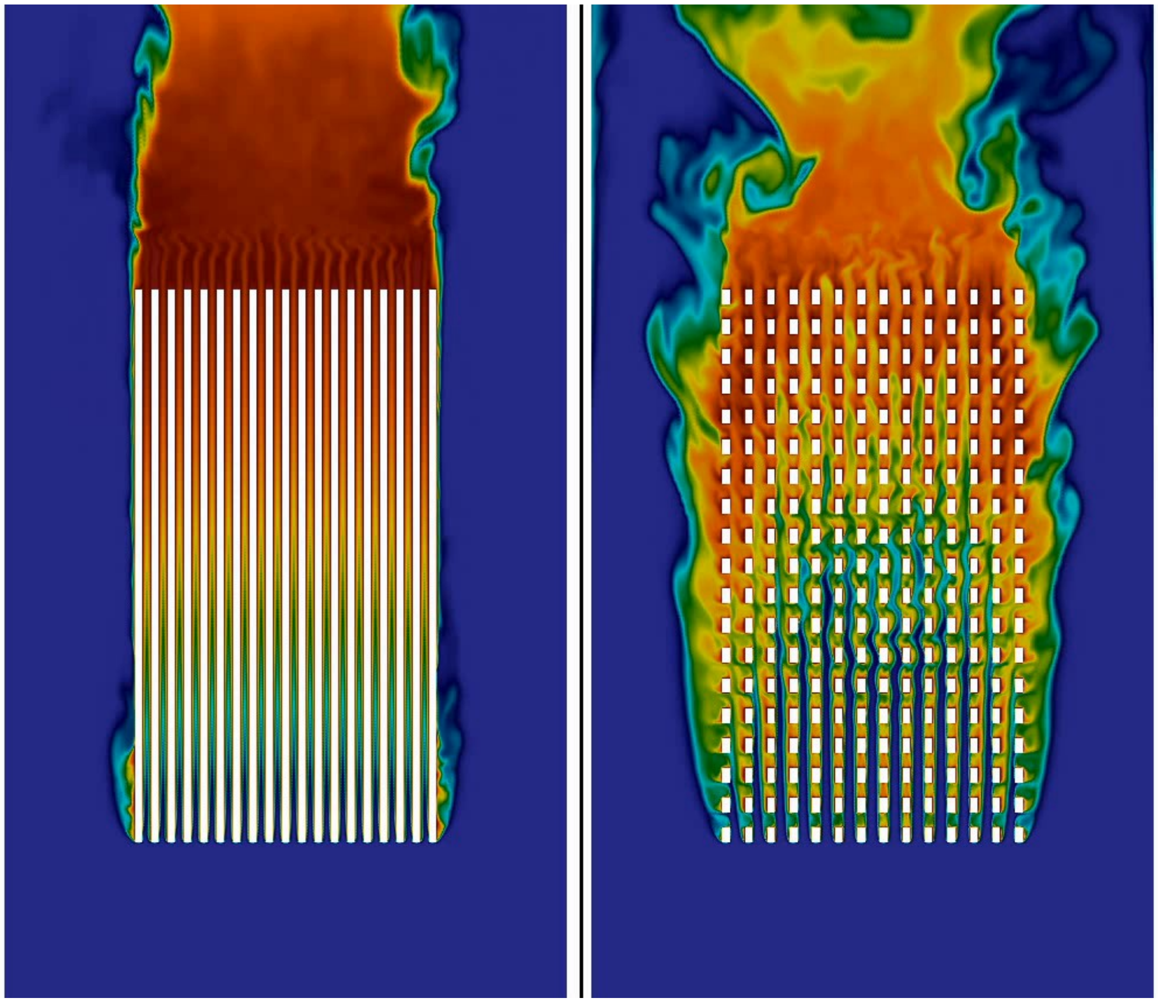


Note: fins

K.Saari, A.Laitinen, K.Kukko, P.Peltonen, V.Vuorinen, J.Partanen (Int.J.Heat and Fluid Flow 2020)

Example: heat exchanger and air cooling for two fin types.

CFD simulation of air temperature from a cross-section of heat exchanger under forced convection: P.Peltonen (2017)



$$c_p \dot{m} \Delta T = \text{Power}$$

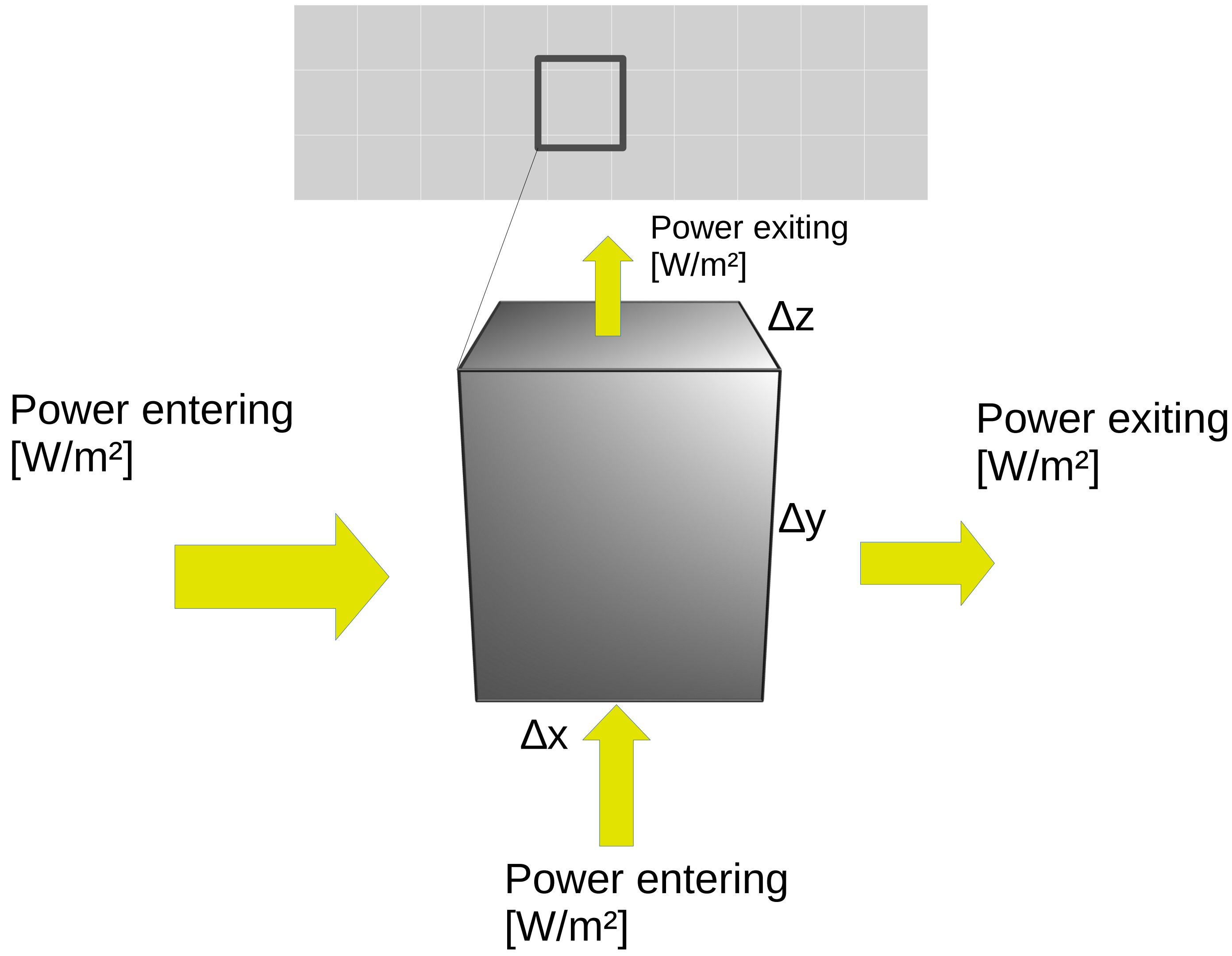
Plate fins   Pin fins

https://www.youtube.com/watch?v=AI1voSK4AJc

Lecture 2.2 Numerical approach: a Matlab solver for the 2d heat equation

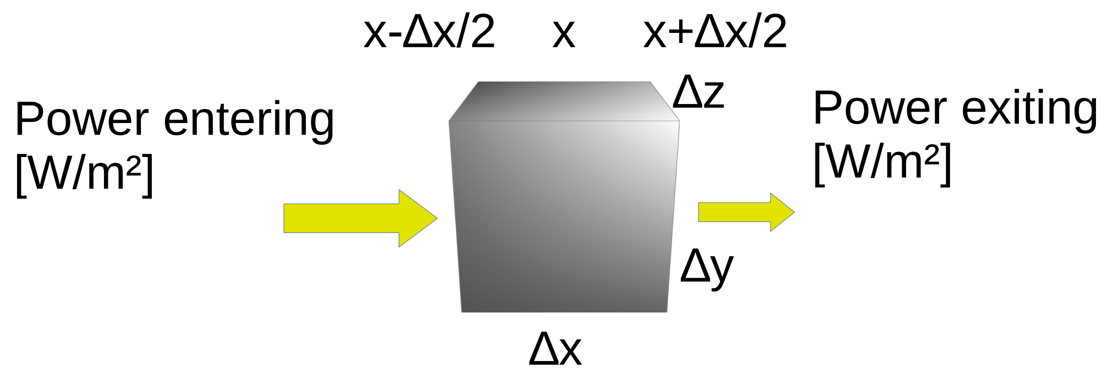
ILO 2: Student can apply Fourier's law and Newton's law in fin theory and thermal resistance context. Further, the student can analyse 2d heat transfer data in Matlab and formulate an energy balance for 2d system.

Consider heat conduction in 2d or 3d object (e.g. metal plate).
Divide the object into small elements and carry out energy balance analysis for 1 of those elements. Assume: no heat losses.



Derivation of heat equation:

Next we apply energy conservation law ("J/s thinking") for a small infinitesimal volume assuming conduction only (e.g.1d metal rod)



Energy increase of element due to heat fluxes in x-direction during Δt (J):

$$\Delta Q_x = [k \frac{\partial T(x+\Delta x/2, y, t)}{\partial x} - k \frac{\partial T(x-\Delta x/2, y, t)}{\partial x}] \Delta y \Delta z \Delta t$$

Energy increase of element due to heat fluxes in y-direction during Δt (J):

$$\Delta Q_y = [k \frac{\partial T(x, y+\Delta y/2, t)}{\partial y} - k \frac{\partial T(x, y-\Delta y/2, t)}{\partial y}] \Delta x \Delta z \Delta t$$

Energy increase of element during Δt (J):

$$\rho c_p \Delta T(x,t) \Delta x \Delta y \Delta z = \Delta Q_x + \Delta Q_y + \Delta Q_z$$

Then: Divide both sides by $\Delta x \Delta y \Delta z \Delta t$ and take the limit when all Δ-variables → 0 → We get the heat equation.

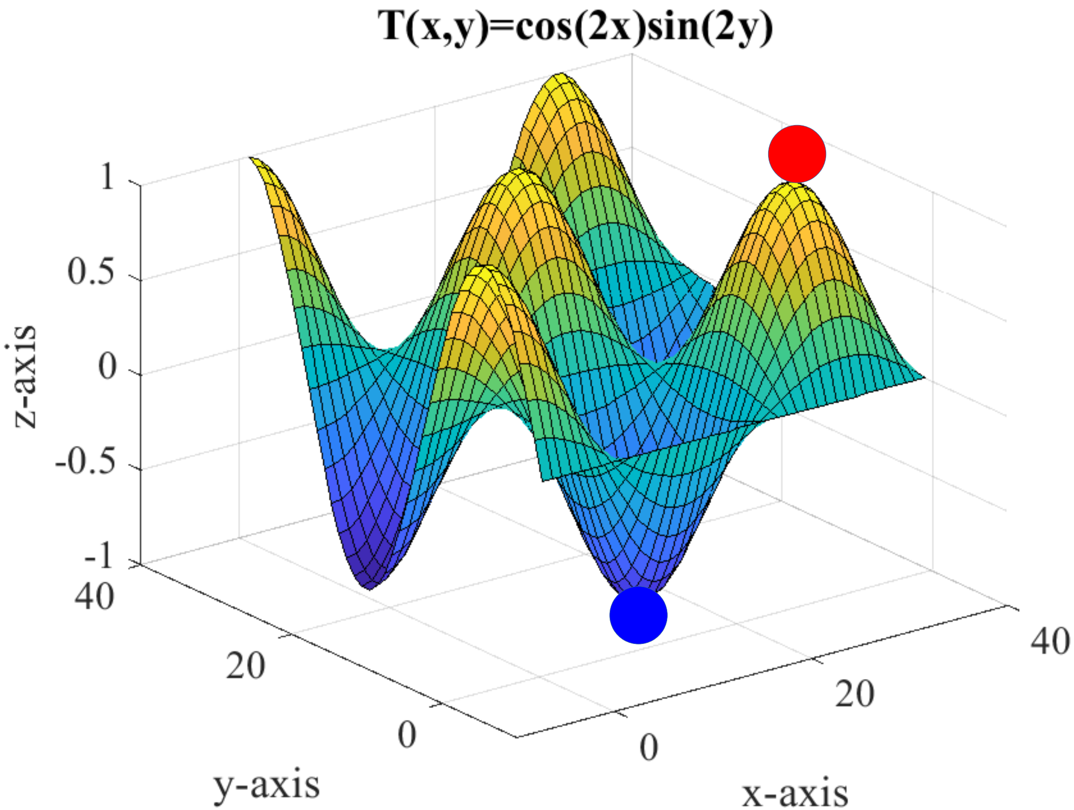
Heat equation in 2d. Well, it is just thermal energy conservation law.

General form

$$\frac{\partial T}{\partial t} = \nabla \cdot \alpha \nabla T$$

Terms opened in 2d (assume α = constant)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \alpha \frac{\partial^2 T}{\partial y^2}$$



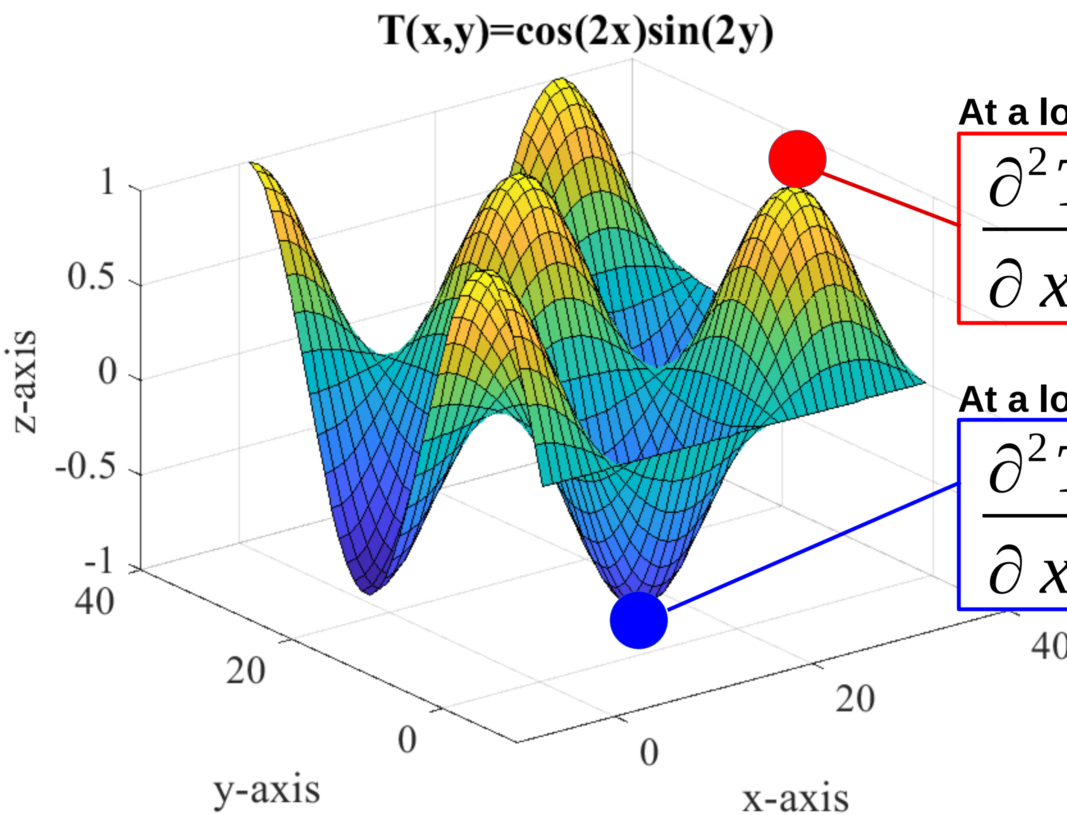
What do the partial derivatives represent? Mathematical interpretation?

General form

$$\frac{\partial T}{\partial t} = \nabla \cdot \alpha \nabla T$$

Terms opened in 2d (assume α = constant)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \alpha \frac{\partial^2 T}{\partial y^2}$$



In conduction problems heat conducts (diffuses) from hot to cold.

$$\frac{\partial T}{\partial t} < 0$$

At a local maximum of a function:

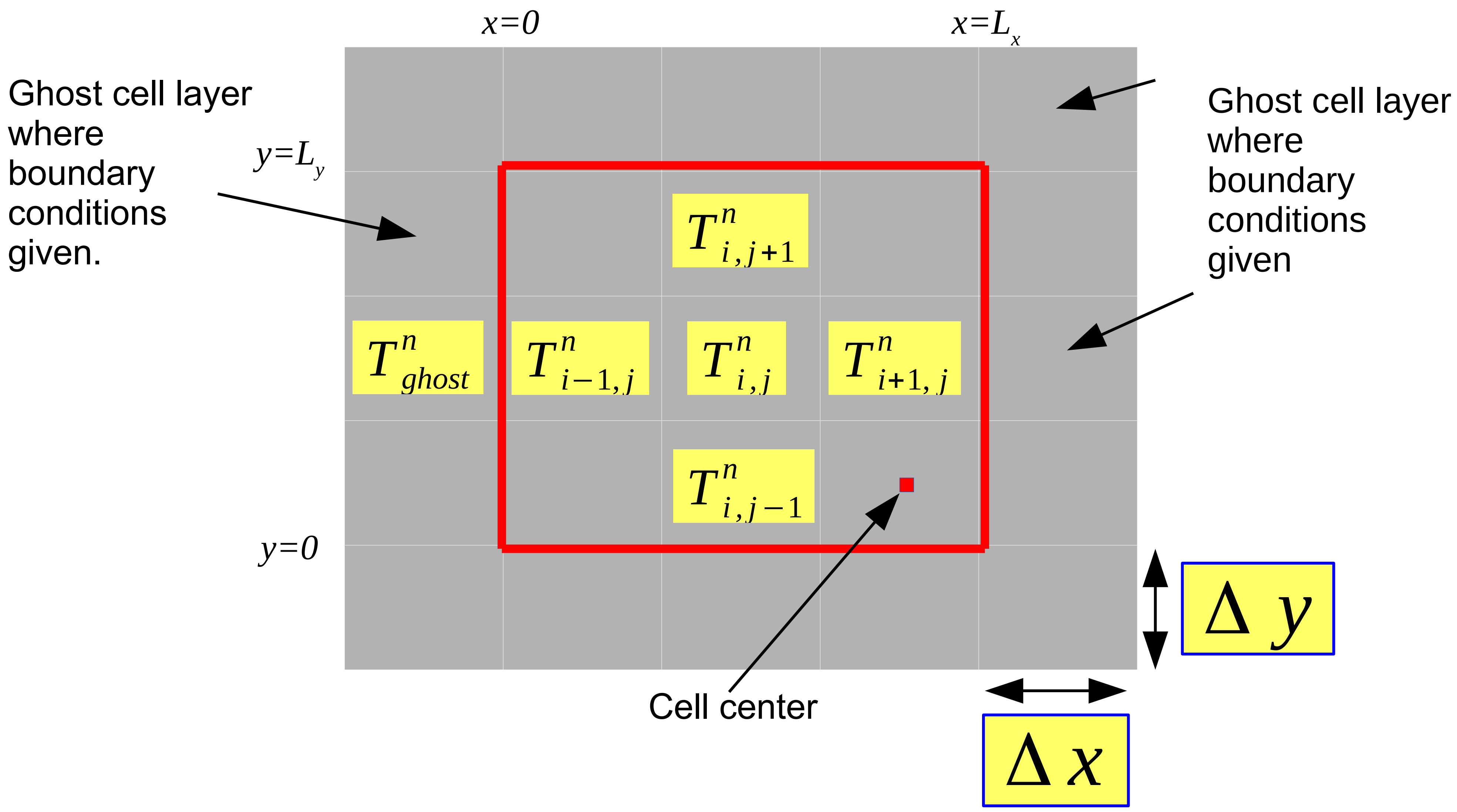
$$\frac{\partial^2 T}{\partial x^2} < 0 \text{ and } \frac{\partial^2 T}{\partial y^2} < 0$$

At a local minimum of a function:

$$\frac{\partial^2 T}{\partial x^2} > 0 \text{ and } \frac{\partial^2 T}{\partial y^2} > 0$$

$$\frac{\partial T}{\partial t} > 0$$

On computers, we can solve heat equation by finite difference methods. We discretize a 2d domain into small elements.



Finite difference discretizations

General form of heat equation

$$\frac{\partial T}{\partial t} = \nabla \cdot \alpha \nabla T$$

Terms opened in 2d

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \alpha \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} \alpha \frac{\partial T}{\partial y}$$

Time derivative in cell (i, j) at timestep n

$$\left(\frac{\partial T}{\partial t} \right)_{i,j}^{n} \approx \frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t}$$

Second space derivatives at cell *(i,j)*

x: $$\left(\frac{\partial^2 T}{\partial x^2} \right)_{i,j}^{n} \approx \frac{T_{i+1,j}^{n} - 2 T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2}$$

y: $$\left(\frac{\partial^2 T}{\partial y^2} \right)_{i,j}^{n} \approx \frac{T_{i,j+1}^{n} - 2 T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2}$$

Update Formula by Explicit Euler Method

$$CFL = \frac{\alpha \Delta t}{\Delta x^2}$$

Explicit Euler timestepping for 2d heat equation:

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \Delta t \, \alpha \frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2} + \Delta t \, \alpha \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2}$$

Which is equal to the “delta” form:

$$\Delta T_{i,j}^{n} = \Delta t \, \alpha \frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2} + \Delta t \, \alpha \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2}$$

Where:

$$\Delta T_{i,j}^{n} = T_{i,j}^{n+1} - T_{i,j}^{n}$$

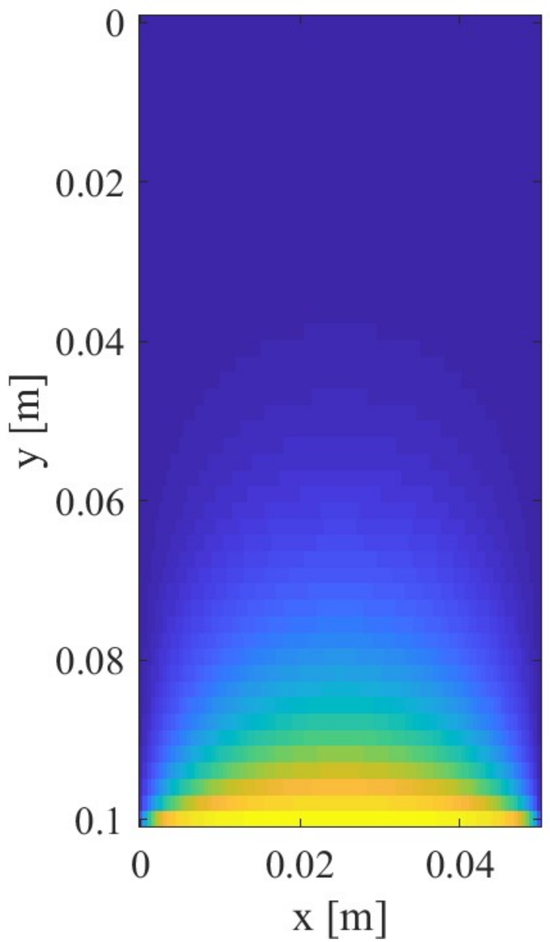
Numerical solution of temperature distribution in a heated 2d metal plate

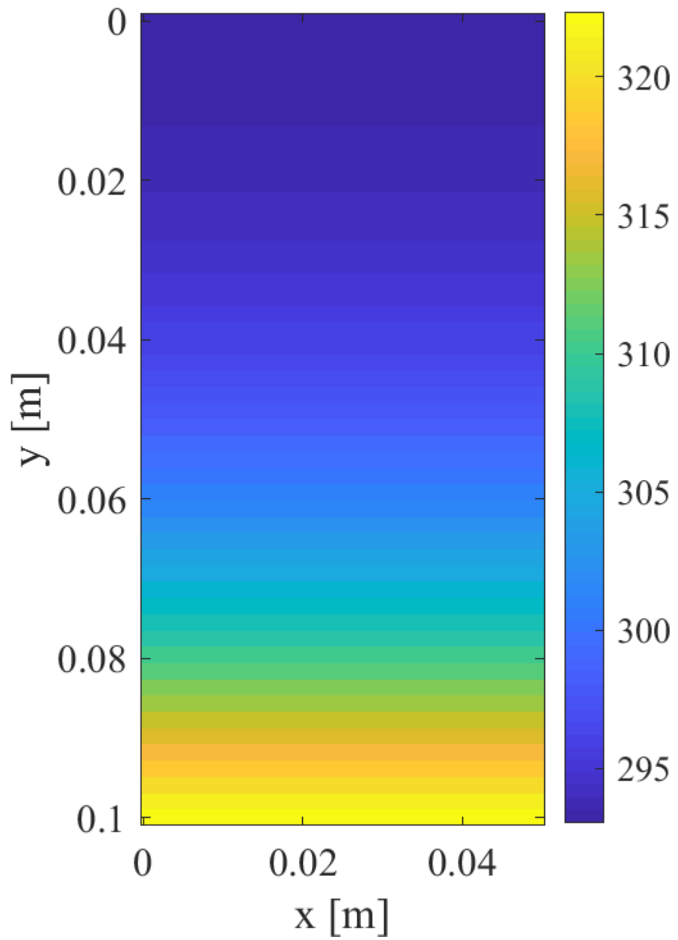
BC 1:
- *Cool sides and top*
- *Hot base*

→ *T=T(x,y,t) (2d)*

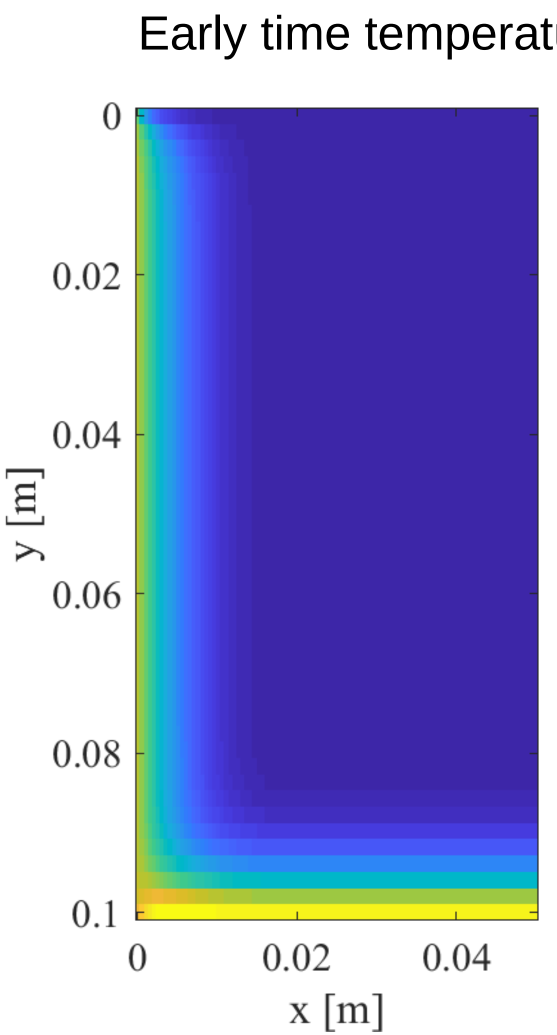
BC 2:
- *Insulated sides*
- *Cool top and hot base*

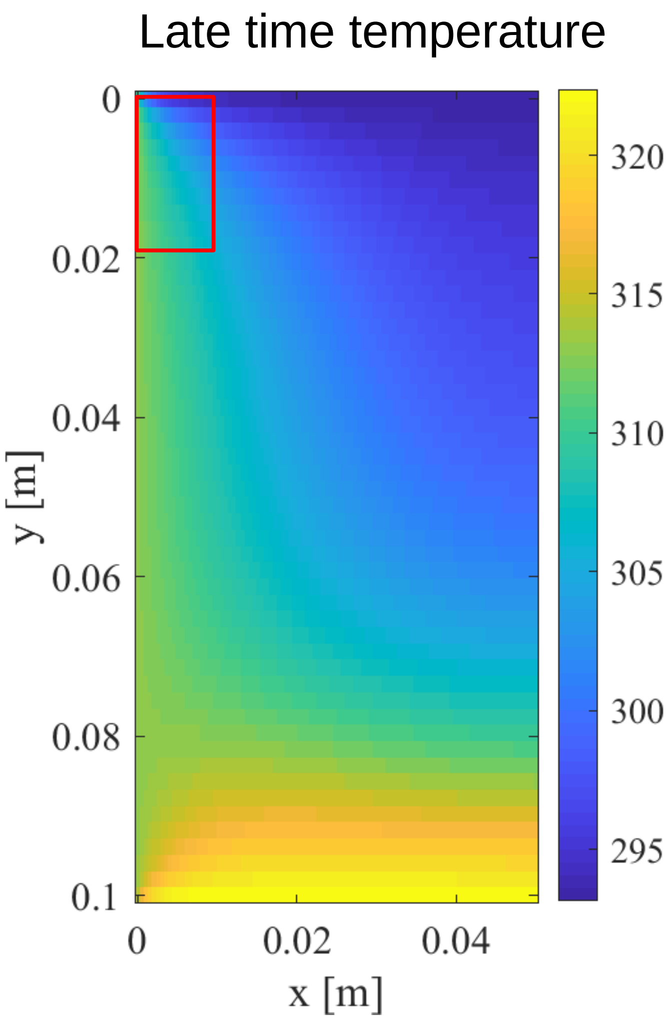
→ *T=T(y,t) (1d)*

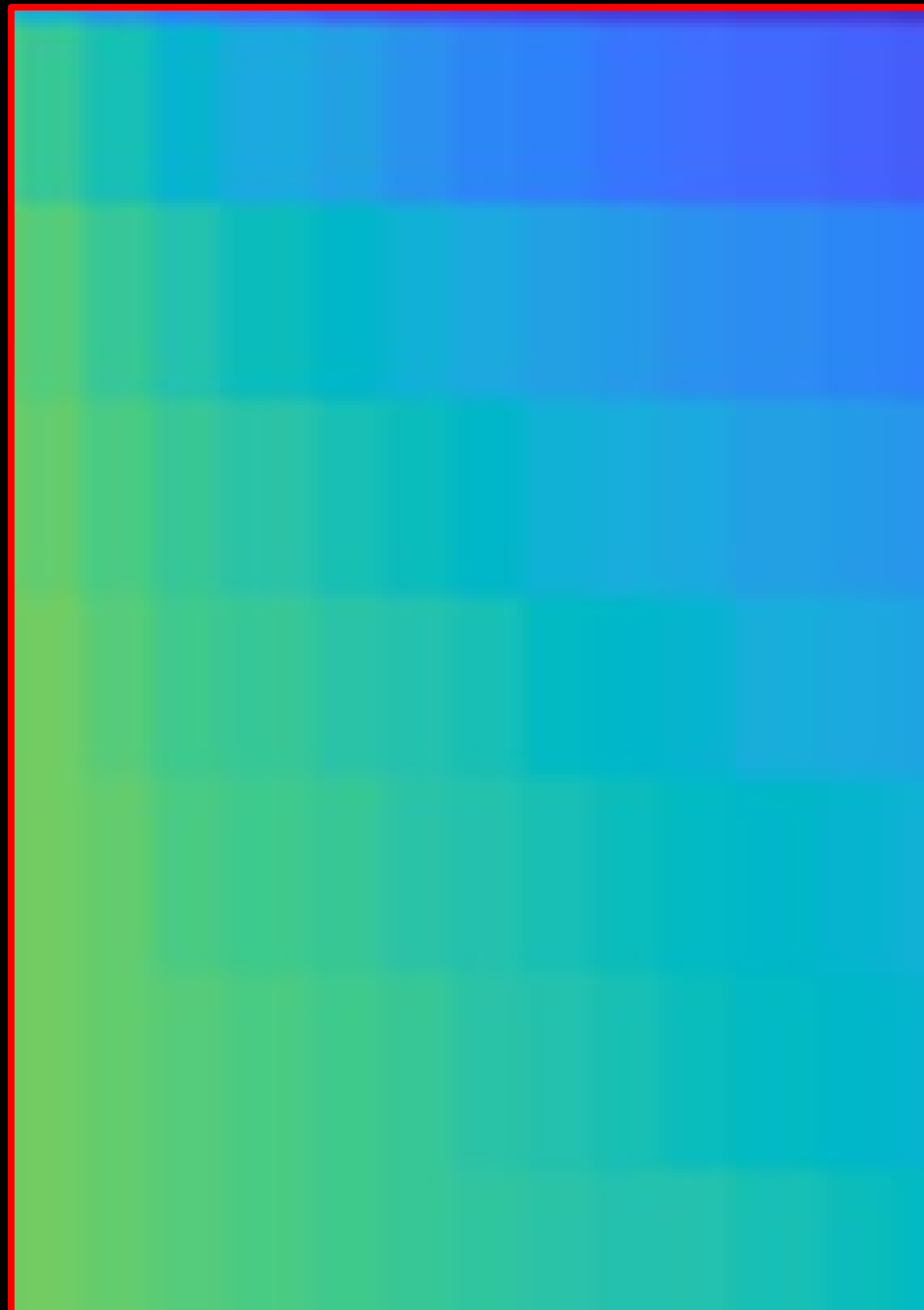




Numerical solution of temperature distribution with two hot, one cold, and one insulated boundary







Zoom to plate
upper left corner

```
>> T(1:20,1:5)

ans =
```

269.8795	276.1205	280.3794	283.3261	285.3968
316.1205	309.8795	305.6206	302.6739	300.6032
314.0774	311.9226	309.8685	308.0042	306.3579
313.6455	312.3545	311.0793	309.8467	308.6758
313.4649	312.5351	311.6104	310.7006	309.8141
313.3662	312.6338	311.9038	311.1806	310.4685
313.3043	312.6957	312.0882	311.4845	310.8868
313.2623	312.7377	312.2138	311.6922	311.1742
313.2321	312.7679	312.3041	311.8418	311.3820
313.2095	312.7905	312.3717	311.9540	311.5380
313.1921	312.8079	312.4239	312.0406	311.6586
313.1784	312.8216	312.4651	312.1091	311.7541
313.1673	312.8327	312.4984	312.1645	311.8312
313.1581	312.8419	312.5257	312.2100	311.8948
313.1505	312.8495	312.5486	312.2481	311.9479
313.1440	312.8560	312.5681	312.2805	311.9932
313.1384	312.8616	312.5850	312.3086	312.0325
313.1334	312.8666	312.5999	312.3334	312.0672
313.1289	312.8711	312.6133	312.3557	312.0984
313.1248	312.8752	312.6256	312.3762	312.1270

```
>> T(1:20,1:5)
```

Ghost cell row of the top side BC

```
ans =
```

269.8795	276.1205	280.3794	283.3261	285.3968
316.1205	309.8795	305.6206	302.6739	300.6032
314.0774	311.9226	309.8685	308.0042	306.3579
313.6455	312.3545	311.0793	309.8467	308.6758
313.4649	312.5351	311.6104	310.7006	309.8141
313.3662	312.6338	311.9038	311.1806	310.4685
313.3043	312.6957	312.0882	311.4845	310.8868
313.2623	312.7377	312.2138	311.6922	311.1742
313.2321	312.7679	312.3041	311.8418	311.3820
313.2095	312.7905	312.3717	311.9540	311.5380
313.1921	312.8079	312.4239	312.0406	311.6586
313.1784	312.8216	312.4651	312.1091	311.7541
313.1673	312.8327	312.4984	312.1645	311.8312
313.1581	312.8419	312.5257	312.2100	311.8948
313.1505	312.8495	312.5486	312.2481	311.9479
313.1440	312.8560	312.5681	312.2805	311.9932
313.1384	312.8616	312.5850	312.3086	312.0325
313.1334	312.8666	312.5999	312.3334	312.0672
313.1289	312.8711	312.6133	312.3557	312.0984
313.1248	312.8752	312.6256	312.3762	312.1270

The corner cell is redundant

Ghost cell column of the left side BC

Note:

1) the two values are different → not insulated Boundary

2) the average of the two values is const. → fixed T_{top} = 293K