1 Mathematical operators

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \tag{1}$$

For vectors:

$$\mathbf{u} = u_x\vec{i} + u_y\vec{j} + u_z\vec{j} \tag{2}$$

$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

(3)

$$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y + \frac{\partial}{\partial z}u_z \tag{4}$$

$$\nabla \cdot \nabla\mathbf{u} = (\frac{\partial^2}{\partial x^2}u_x + \frac{\partial^2}{\partial y^2}u_x + \frac{\partial^2}{\partial z^2}u_x)\vec{i} + (\frac{\partial^2}{\partial x^2}u_y + \frac{\partial^2}{\partial y^2}u_y + \frac{\partial^2}{\partial z^2}u_y)\vec{j} + (\frac{\partial^2}{\partial x^2}u_z + \frac{\partial^2}{\partial y^2}u_z + \frac{\partial^2}{\partial z^2}u_z)\vec{k} \tag{5}$$

$$\mathbf{u} \cdot \nabla\mathbf{u} = (u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z})\vec{i} + (u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z})\vec{j} + (u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z})\vec{k}$$

(6)

For scalars:

$$\nabla T = \frac{\partial}{\partial x}T\vec{i} + \frac{\partial}{\partial y}T\vec{j} + \frac{\partial}{\partial z}T\vec{k} \tag{7}$$

$$\nabla \cdot \nabla T = \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T + \frac{\partial^2}{\partial z^2}T \tag{8}$$

2 Basic Principles

Change in internal energy is

$$\Delta Q = mc_p \Delta T \tag{9}$$

where $[Q] = J$, and the rate of heat transfer for an object with uniform mass is

$$q = mc_p \frac{\partial T}{\partial t} \tag{10}$$

where $[q] = W$, $[m] = kg$ and $[c_p] = J/kgK$ is the specific heat. Newton's law of cooling (conduction) is

$$q = hA(T_s - T_\infty) \tag{11}$$

where $[h] = W/m^2K$ is the heat transfer coefficient, T_s is the surface temperature of the object, T_∞ is the temperature far away from the object and $[A] = m^2$ is the (cross-sectional) surface area. In case of e.g. cooling of an object, to ensure conservation of energy, we must have

$$q_1 = -q_2 \tag{12}$$

i.e.

$$mc_p \frac{\partial T}{\partial t} = -hA(T_s - T_\infty) \tag{13}$$

Note that it is common to replace $T_s - T_\infty$ with θ.

3 Heat conduction, Incropera Ch. 3, 4

Heat conduction (Fourier's law) is

$$q'' = -k\nabla T \tag{14}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, $[k] = W/mK$ is thermal conductivity and $[q''] = W/m^2$ is heat flux. The relation between rate of heat transfer and heat flux is

$$q = q''A \tag{15}$$

At steady state and in 1D, Fourier's law is

$$q'' = -k\frac{\Delta T}{\Delta x} \tag{16}$$

and the rate of heat transfer may be reformulated as

$$q = \frac{\Delta T}{R} \tag{17}$$

where $[R] = K/W$ is thermal resistance. Thermal resistance for conduction is

$$R = \frac{\Delta x}{Ak} \tag{18}$$

and for convection

$$R = \frac{1}{hA} \tag{19}$$

total thermal resistance for parallel conduction is

$$\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2} + ... \tag{20}$$

and in series

$$R_{tot} = R_1 + R_2 + ... \tag{21}$$

In radial direction we have

$$q_r = -k(2\pi r L)\frac{dT}{dr} \tag{22}$$

which at steady state is

$$q_r = \frac{2\pi L k \Delta T}{\ln(r_2/r_1)} \tag{23}$$

where L is the length of the pipe. Conduction thermal resistance is of form

$$R_{cond} = \frac{\ln(r_2/r_1)}{2\pi L k} \tag{24}$$

and the convective resistance is

$$R_{conv} = \frac{1}{h 2\pi r L}. \tag{25}$$

Heat diffusion is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{26}$$

where $\nabla^2 = (\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2})$ and $[\alpha] = m^2/s$ is thermal diffusivity. Thermal diffusivity may be also written as

$$\alpha = \frac{k}{\rho c_p} \tag{27}$$

where $[\rho] = kg/m^3$ is density. At steady state, heat diffusion reduces to

$$\nabla^2 T = 0 \tag{28}$$

4 Internal flow, Incropera Ch. 6, 8

In convection heat transfer, the concept of a control volume becomes an important tool. The control volume is a somewhat arbitrary volume which contains the heat transfer system over which the heat balance is calculated. What happens inside the control volume in detail is not of high importance if we are calculating global quantities, such as the rate of heat transfer. The heat balance over the control volume at steady state must always be zero to fulfill energy conservation. In internal flow cases the control volume is often the dimensions of the pipe, which may be quite intuitive.

Fluid flow is governed with the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{29}$$

and the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} + \mathbf{g} \tag{30}$$

while the heat flow is governed with the convection-diffusion equation

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \alpha \nabla^2 T + q''' \tag{31}$$

where q''' is a volumetric heat source.

The rate of heat transfer in internal flow is

$$q = \dot{m} c_p \Delta T \tag{32}$$

where $[\dot{m}]$ is the mass flow rate of the fluid and $\Delta T = T_{out} - T_{in}$ is the temperature difference between the inlet and the outlet. Now, the temperature difference in Newton's law of cooling

$(T_s - T_\infty)$ changes as the temperature of the fluid (T_∞) changes, and hence, we must take it into consideration by using a logarithmic mean temperature difference

$$\Delta T_{lm} = \frac{(T_s - T_{in}) - (T_s - T_{out})}{\ln(\frac{T_s - T_{in}}{T_s - T_{out}})} \tag{33}$$

or written in a simpler form

$$\Delta \theta_{lm} = \frac{\theta_{in} - \theta_{out}}{\ln(\theta_{in}/\theta_{out})} \tag{34}$$

Hence, the newton's law of cooling can be written as

$$q = hA\Delta T_{lm} \tag{35}$$

and the heat balance is

$$\dot{m}c_p(T_{out} - T_{in}) = hA\Delta T_{lm} \tag{36}$$

where A is the cross-sectional surface area of the inside of the pipe. The heat transfer coefficient, h, is commonly evaluated from heat transfer correlations for the dimensionless Nusselt number Nu. The Nusselt number describes the ratio between convective and conductive heat transfer in a fluid, and may be written as

$$Nu_L = \frac{hL}{k} \tag{37}$$

where L is some characteristic length that depends on the fluid flow situation. For internal flow, the characteristic length is most commonly D_h, i.e. the hydraulic diameter. The subscript in Nu often describes what characteristic length is used. The ratio of the thermal boundary layer to the hydrodynamic boundary layer is described with the Prandtl number

$$Pr = \frac{c_p \mu}{k} = \frac{\nu}{\alpha} \tag{38}$$

which can be calculated with the given material properties. Tabulated values may be used as well.

Another important dimensionless number here is the Reynolds number Re, which describes whether the fluid flow is laminar or turbulent

$$Re_L = \frac{UL}{\nu} \tag{39}$$

where U is the bulk velocity of the fluid, ν is the kinematic viscosity of the fluid and L is again some characteristic length. The critical Reynolds number, i.e. the number at which the flow is no longer laminar and becomes turbulent changes according to the fluid flow situation. For internal pipe flow the critical Reynolds number is commonly $Re_{crit} = 2300$.

More information on the different heat transfer correlations can be found in Incropera chapter 8.

5 External flow, Incropera Ch. 7

The same equations apply here as in internal flow with some changes in the characteristic lengths, the cross-sectional surface areas, critical Reynolds numbers etc. The concept of the control volume becomes more important in external flow cases, since the heat balance is altered depending on the dimensions of the control volume. This is due to e.g. mass flow rate and the total rate of heat transfer being extensive quantities. Hence, the dimensions of the control volume should be properly defined. The control volume should be applied either over an entire system, or over some repetitive structure of the geometry, such as, a single row in a bank of tubes with N rows, or 1 meter in depth of an infinitely wide plate.

More information on the different heat transfer correlations can be found in Incropera chapter 7.

6 Natural convection, Incropera Ch. 9

In natural convection, the convection of the fluid arises from density differences in the fluid. A warmer fluid has a lower density compared to a cooler fluid and due to gravity, it begins to rise up. In natural convection, to evaluate the magnitude of the fluid flow, instead of the Reynolds number, we use the dimensionless Rayleigh number Ra,

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha} \tag{40}$$

or Grashof number Gr,

$$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} \tag{41}$$

where $[g] = m/s^2$ is the gravitational constant, $[\beta] = 1/K$ is the volumetric thermal expansion coefficient and again, L is some characteristic length. For an ideal gas the volumetric thermal expansion coefficient is $\beta = 1/T$. Note that to evaluate $T_s - T_\infty$, it is common to use average estimations instead of accurate values.

More information on the different heat transfer correlations can be found in Incropera chapter 9.

7 Radiation, Incropera Ch. 12, 13

Stefan-Boltzmann law for black body radiation is

$$q'' = \sigma T^4 \tag{42}$$

where $\sigma = 5.67 \cdot 10^{-8} W/m^2K^4$ is the Stefan-Boltzmann coefficient. Since real bodies do not absorb all radiation, an emissivity ϵ (< 1) is used to evaluate radiation heat transfer

$$q'' = \epsilon\sigma T^4 \tag{43}$$

The value of the emissivity depends on the surface material (tabulated values should be used for calculations). Since all bodies with temperature above zero emit thermal radiation, the radiation heat transfer between two bodies is

$$q'' = \epsilon\sigma(T_1^4 - T_2^4) \tag{44}$$