Problem 7.1: KKT Conditions for Equality Constrained Problems

Let $X \subset \mathbb{R}^n$ be a nonempty open set, and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Moreover, let $g_i : \mathbb{R}^n \to \mathbb{R}$ be differentiable for all $i = 1, \dots, m$, and let $h_i : \mathbb{R}^n \to \mathbb{R}$ be differentiable for all $i = 1, \dots, l$. Consider the following optimization problem P:

$$\begin{aligned}
(P) : \ & \text{min. } f(x) \\
\text{subject to: } & g_i(x) \leq 0, && i = 1, \dots, m \\
& h_i(x) = 0, && i = 1, \dots, l \\
& x \in X
\end{aligned}$$

Let $\overline{x}$ be a feasible solution to P, and let $I = \{i : g_i(\overline{x}) = 0\}$ be the index set of *active* inequality constraints. Also, let $\nabla g_i(\overline{x})$ for $i \in I$ and $\nabla h_i(\overline{x})$ for $i = 1, \dots, l$ be linearly independent (to enforce constraint qualification). Derive KKT conditions for the problem P.

Hint: Notice that $h_i(x) = 0$ can be equivalently replaced by the two inequalities

$$h_i(x) \leq 0 \text{ and } -h_i(x) \leq 0.$$

Solution.

To simplify notation, let us first define the following:

$$\tilde{g} = \begin{cases} g_i, & i = 1, \dots, m \\ h_{i-m}, & i = m+1, \dots, m+l \\ -h_{i-m-l}, & i = m+l+1, \dots, m+2l, \end{cases}$$

Using $\tilde{g}$, we can rewrite the problem P as

$$\begin{aligned}
(P) : \ & \text{min. } f(x) && (1) \\
\text{subject to: } & \tilde{g}_i(x) \leq 0, \ i = 1, \dots, m+2l && (2) \\
& x \in X && (3)
\end{aligned}$$

The KKT conditions for a feasible solution $\overline{x}$ to the problem (1) – (3) are given by

$$\begin{aligned}
& \nabla f(\overline{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\overline{x}) + \sum_{i=m+1}^{m+l} u_i \nabla h_{i-m}(\overline{x}) - \sum_{i=m+l+1}^{m+2l} u_i \nabla h_{i-m-l}(\overline{x}) = 0 && (4) \\
& u_i g_i(\overline{x}) = 0, \ i = 1, \dots, m && (5) \\
& u_i h_{i-m}(\overline{x}) = 0, \ i = m+1, \dots, m+l && (6) \\
& -u_i h_{i-m-l}(\overline{x}) = 0, \ i = m+l+1, \dots, m+2l && (7) \\
& u_i \geq 0, \ i = 1, \dots, m+2l && (8)
\end{aligned}$$

Notice that $i \in I$ for $i = m+1, \dots, m+2l$, thus rendering (6) and (7) redundant. Letting $v_i = u_{m+i} - u_{m+l+i}$ for $i = 1, \dots, l$ implies that $v_i \in \mathbb{R}$ for $i = 1, \dots, l$. Combining these, we can finally rewrite the KKT conditions (4) – (8) as

$$\begin{aligned}
& \nabla f(\overline{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\overline{x}) + \sum_{i=i}^{l} v_i \nabla h_i(\overline{x}) = 0 \\
& u_i g_i(\overline{x}) = 0, \ i = 1, \dots, m \\
& u_i \geq 0, \ i = 1, \dots, m.
\end{aligned}$$

Problem 7.2: KKT Transformation of a Bilevel Optimization Problem

Consider the following *bilevel* optimization problem:

$$\min_{x} \; c_1^\top x + c_2^\top y \tag{9}$$
$$\text{subject to: } Ax + By \leq \alpha \tag{10}$$
$$y \in \operatorname*{argmin}_{y} \;\; c_3^\top y \tag{11}$$
$$\text{subject to: } \; Dx + Ey \leq \beta \tag{12}$$

In problem (9) – (12), we seek an optimal value of x knowing that y, which minimizes another optimization problem, depends on the value of x. This is a way of modeling hierarchical decision problems such as Stackelberg competition.

Reformulate the problem (9) – (12) by replacing the constraints (11) – (12) that form the inner optimization problem:

$$\begin{aligned} y \in \operatorname*{argmin}_{y} \;\; & c_3^\top y \\ \text{subject to: } \;\; & Dx + Ey \leq \beta \end{aligned}$$

with the KKT optimality conditions of this problem. You can assume that $\beta \in \mathbb{R}^m$, which implies that (12) has $i = 1, \ldots, m$ inequality constraints. Is the resulting problem convex? Justify your answer.

Hint: You can write the constraint (12) as

$$d_i x + e_i y \leq \beta_i, \quad i = 1, \ldots, m$$

where d_i and e_i correspond to the ith rows of the matrices D and E, respectively, and β_i is the ith element of the vector $\beta \in \mathbb{R}^m$.

Solution.

The reformulated problem becomes:

$$\begin{aligned} \min_{x,y,u} \;\; & c_1^\top x + c_2^\top y \\ \text{subject to: } \;\; & Ax + By \leq \alpha \\ & c_3 + E^\top u = 0 && \text{(dual feasibility 1)} \\ & Dx + Ey - \beta \leq 0 && \text{(primal feasibility)} \\ & u_i(d_i x + e_i y - \beta_i) = 0, \; i = 1, \ldots, m && \text{(complementary slackness)} \\ & u \geq 0 && \text{(dual feasibility 2)} \end{aligned}$$

or

$$\begin{aligned} \min_{x,y,u,s} \;\; & c_1^\top x + c_2^\top y \\ \text{subject to: } \;\; & Ax + By \leq \alpha \\ & c_3 + E^\top u = 0 && \text{(dual feasibility 1)} \\ & Dx + Ey + s = \beta && \text{(primal feasibility)} \\ & u_i s_i = 0, \; i = 1, \ldots, m && \text{(complementary slackness)} \\ & u \geq 0 && \text{(dual feasibility 2)} \end{aligned}$$

The problem is not convex due to the bilinear constraints arising from complementary slackness.

Problem 7.3: Example of a Bilevel Transformation

Consider the following bilevel optimization problem:

$$
\begin{aligned}
\min_{x}.\ & x - 4y && (13)\\
\text{subject to: } & x \geq 0 && (14)\\
& y \in \operatorname*{argmin}_{y} y && (15)\\
& \text{subject to: } -x - y \leq -3 && (16)\\
& \qquad -2x + y \leq 0 && (17)\\
& \qquad 2x + y \leq 12 && (18)\\
& \qquad -3x + 2y \leq -4 && (19)\\
& \qquad y \geq 0 && (20)
\end{aligned}
$$

Reformulate the problem (13) – (20) by replacing the constraints (15) – (20) that form the inner optimization problem:

$$
\begin{aligned}
y \in \operatorname*{argmin}_{y} \ & y\\
\text{subject to: } & -x - y \leq -3\\
& -2x + y \leq 0\\
& 2x + y \leq 12\\
& -3x + 2y \leq -4\\
& y \geq 0
\end{aligned}
$$

with the KKT conditions of this problem. Try to model and solve the reformulated problem with `Julia` using `JuMP`. One locally optimal solution for the problem is $(x, y) = (2, 1)$ with objective value $f(x, y) = x - 4y = -2$. Can you find this local optimum by trying different initial (starting) values for the different variables? Is the reformulated problem convex? Justify your answer.

Solution.

The reformulated problem is of the form

$$
\begin{aligned}
\min_{x,y,u}.\ & x - 4y\\
\text{subject to: } & 1 - u_1 + u_2 + u_3 + 2u_4 = 0\\
& u_1(-x - y + 3) = 0\\
& u_2(-2x + y) = 0\\
& u_3(2x + y - 12) = 0\\
& u_4(-3x + 2y + 4) = 0\\
& -x - y \leq -3\\
& -2x + y \leq 0\\
& 2x + y \leq 12\\
& -3x + 2y \leq -4\\
& y \geq 0\\
& x \geq 0\\
& u_1, \dots, u_4 \geq 0
\end{aligned}
$$

See the Julia code. The problem is not convex due to the complementary slackness conditions.