Linear Programming Modelling

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Optimization modelling

- The problem is defined in *implicitly* terms of
  - an **Objective function** to minimize of maximize
  - by choosing optimal values for **decision variables**
  - subject to **constraints**
- Optimization software solves the problem automatically
  - This approach is a dramatically different from *explicit* (simulation) models where the result is obtained by applying some formulas in given order
- Most common optimization model types:
  - Linear Programming (LP) problem
  - Mixed Integer Linear Programming (MILP) problem

Optimization problem example

- Sample problem with two variables

$$\min x_1^2 + x_2^2$$

s.t.

$$x_1 + x_2 \geq 3$$

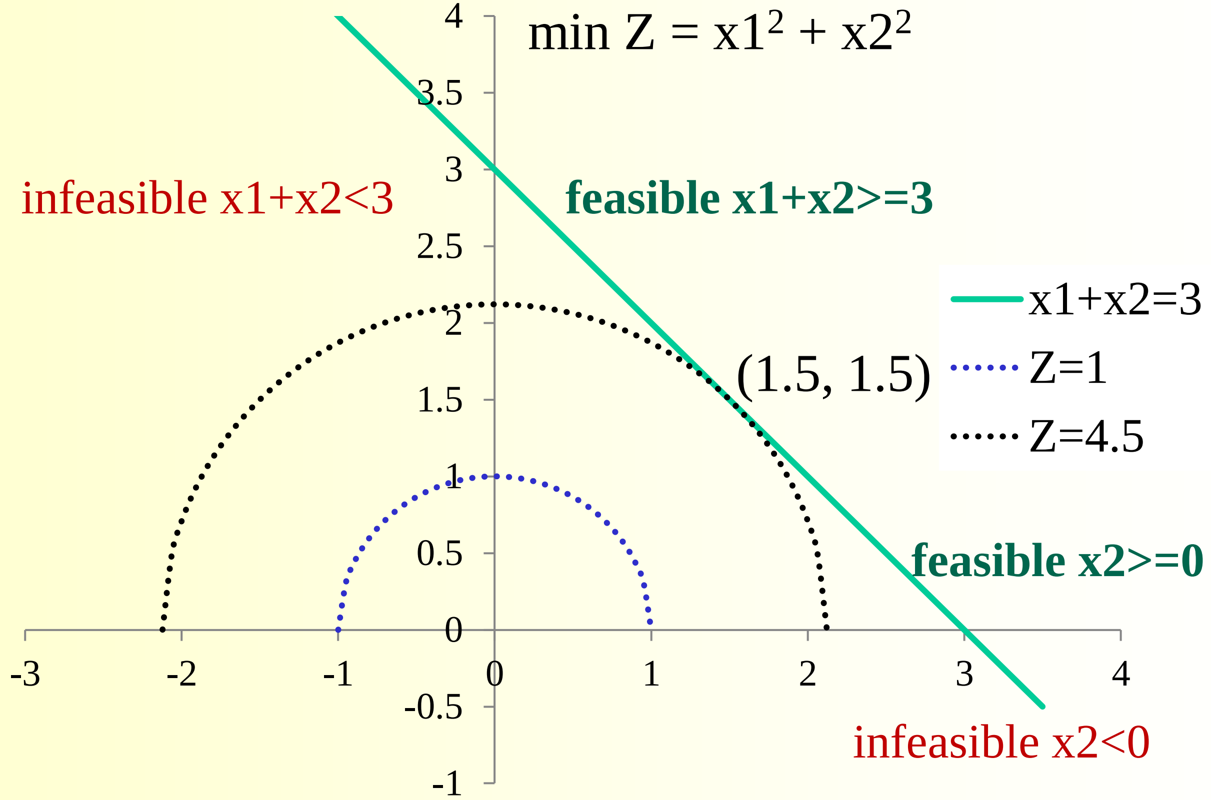
$$x_2 \geq 0$$

$$x_1, x_2 \in R$$

- In a two-dimensional case the problem can be illustrated and solved graphically
  - Constraints define the feasible region in the plane
  - Level curves of the objective function show the *height of the terrain*

Optimization problem example

- **Constraints** define the feasible region in the plane
- **Level curves** of the objective function show the height of the terrain



General mathematical optimization problem

Find values for ***decision variables* x** that minimize or maximize the ***objective function*** f(**x**) subject to ***constraints***:

min $f(\mathbf{x})$	(objective function)
subject to	
$\mathbf{h}(\mathbf{x}) = \mathbf{0}$	(vector of equality constraints)
$\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$	(vector of inequality constraints)
$\mathbf{x} \in R^n$ (or $\mathbf{x} \in N^n$)	(vector of decision variables)

If domain of **x** is R^n, it is a ***continuous optimization problem***

If all x_i are integers, it is an ***integer optimization problem***

A ***mixed integer problem*** contains both integer and real x_i

Properties of optimization problems

- Consider general problem min (max) $f(\mathbf{x})$ s.t. $\mathbf{x} \in X$
- A particular solution $\mathbf{x} = \mathbf{x}^*$ is
  - ***feasible*** if it satisfies all constraints (i.e. $\mathbf{x}^* \in X$)
  - ***infeasible*** if it does not satisfy all constraints
  - ***optimal*** if it is feasible and minimizes (maximizes) $f(\mathbf{x})$
- The ***problem*** is
  - ***feasible*** if at least one feasible solution exists
  - ***infeasible*** if no feasible solution exists
  - ***unbounded*** if infinitely good feasible solutions exist
- The problem can have
  - no optimal solutions: when the problem is infeasible or unbounded
  - one unique optimal solution
  - multiple (equally good) optimal solutions

Optimization model transformations

- Transformations
  - $\min f(\mathbf{x}) = -\max -f(\mathbf{x})$
  - $\max f(\mathbf{x}) = -\min -f(\mathbf{x})$
  - $g(\mathbf{x}) \leq 0 \Leftrightarrow -g(\mathbf{x}) \geq 0$
  - $g(\mathbf{x}) = 0 \Leftrightarrow g(\mathbf{x}) \leq 0 \wedge g(\mathbf{x}) \geq 0$
  - $g(\mathbf{x}) \leq 0 \Leftrightarrow g(\mathbf{x}) + s^2 = 0$ where s is an unconstrained real variable
- Constrained problem can be transformed into unconstrained by augmenting objective with a penalty term, i.e. a ***barrier function***
  - $\min f(\mathbf{x})$ s.t. $g(\mathbf{x}) \leq 0 \Leftrightarrow \min f(\mathbf{x})+M\cdot\max\{\mathbf{g}(\mathbf{x}),0\}$
    - M is a big positive number

Optimization model types

- Depending on the structure of the objective function and constraints, optimization models can be classified in different ways
  - Single variable and multiple variables
  - Continuous, discrete or mixed integer problems
    - Decision variables are continuous, binary (0/1), general integers, or mixed
    - Integer programming, mixed integer programming
  - Unconstrained and constrained problems
  - Convex and non-convex problems
    - Linear, quadratic and nonlinear problems
  - Single objective and multi-objective problems
    - $\mathbf{f}(\mathbf{x})$ is a vector of objective functions

Optimization model types – Exampes

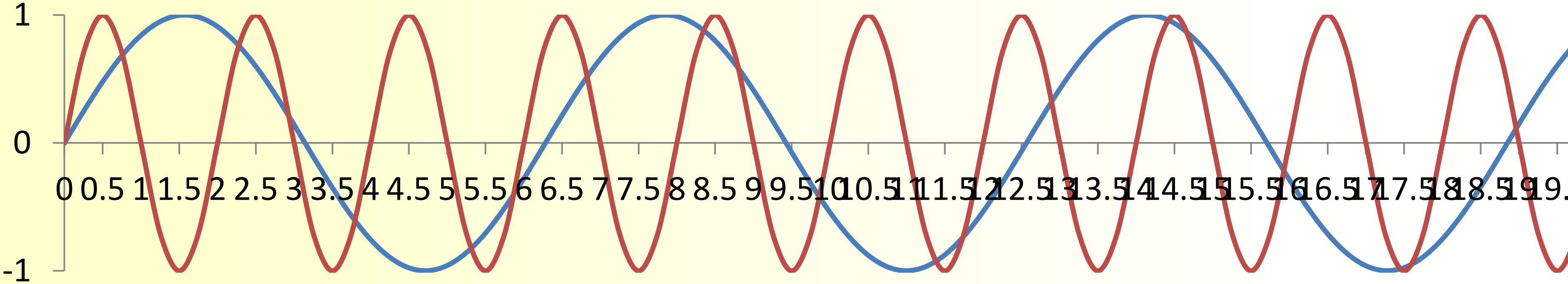
- Sizing of ground source heat pump
  - Single objective (minimize life-cycle costs)
  - Single continuous variable (size of pump)
  - Constrained non-linear convex problem
- Unit commitment of power plants
  - Single objective (maximize profit)
  - Multiple variables of mixed types
  - Constrained non-convex problem
- Investment in new production technology
  - Multiple objectives (economic, environmental, policy, …)
  - Multiple discrete (binary) variables
  - Constrained or unconstrained problem

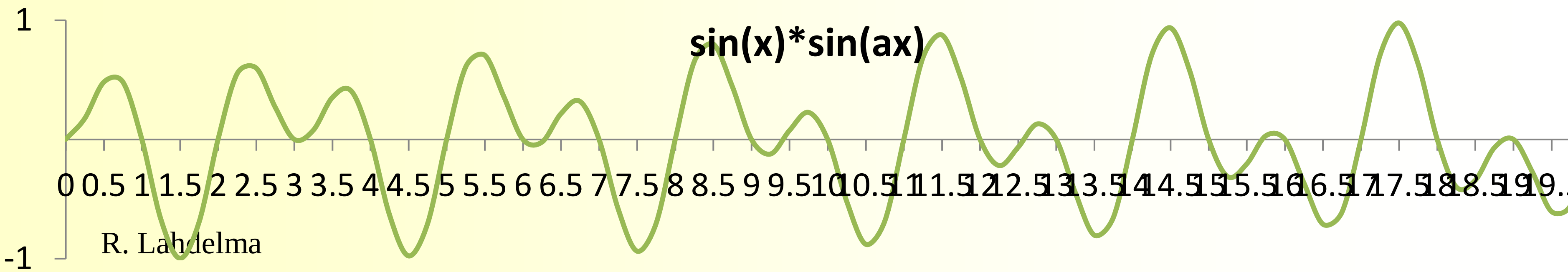
Solving optimization problems

- To solve problems it is necessary to understand the different problem types and their properties
  - There is no universal way to find the optimum or even a feasible solution to an arbitrary problem
  - Different solution algorithms are required for different problem types
- **Most important is to determine if the optimization problem is convex or not!**
  - Convex problem = minimize convex objective function in a convex region
  - Convex problem: relatively easy
  - Non-convex problem: potentially very hard

Impossible to solve non-convex model

- Consider max/min f(x)=sin(x)*sin(ax)
  - Each factor has max/min at +1/-1
  - If the peaks and valleys coincide then f(x)=+1 or -1
  - If a is chosen properly, peaks and valleys never meet
  - No optimum, values approaching +1/-1





Real-life optimization problems

- A real-life model differs from theoretical models in several aspects
  - Normally the problem is never unbounded
  - The existence of a feasible solution can often be verified intuitively
  - Often many model parameters are uncertain or imprecise
  - It is not necessary to find the true optimum – a near-optimal solution and sometimes even a reasonably good solution may suffice

LP and MILP modelling

- ***Linear Programming*** and ***Mixed Integer Linear Programming*** are most commonly used approaches for practical problems because
  - the modelling techniques are very versatile and flexible
  - efficient and reliable solvers exist for these problems
- Arbitrary convex optimization problems can be approximated by LP models
- Many non-convex optimization problems can be approximated by MILP models

LP modelling

- By far, the most commonly used optimization modelling technique
  - Applicable for a wide class of different problems
  - Easy to formulate
  - Easy to understand
  - Very large models can be solved efficiently
  - Interpretation of results and various sensitivity analyses are (relatively) easy
- Many energy optimization problems can be represented as LP models
  - Why can LP modelling not always be used?

Applicability of LP models

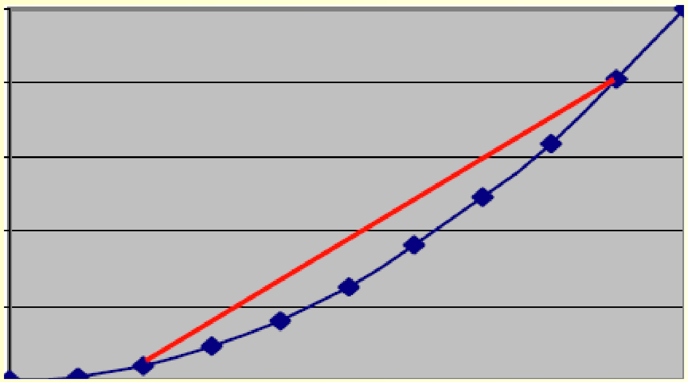
- LP models work only in convex problems
  - The **minimization problem is convex** when:
    - The minimized objective function is **convex**
    - The feasible region is **convex**
  - The **maximization problem is convex** when:
    - The maximized objective function is **concave**
    - The feasible region is **convex**
  - An LP model is a piecewise linear convex model
- How can non-convex problems be modelled?

Convex optimization problem

- A convex optimization problem is of form min f(x); s.t. $\mathbf{x} \in X$
  - where f() is a convex function and
  - X is a convex set
- Similarly max f($\mathbf{x}$) s.t. $\mathbf{x} \in X$ where f() is a concave function is a convex optimization problem
- The feasible region X is a convex set when
  - functions in inequality constraints $g(\mathbf{x}) \leq 0$ are convex and
  - functions in equality constraints $h(\mathbf{x}) = 0$ are linear.

Convex and concave functions

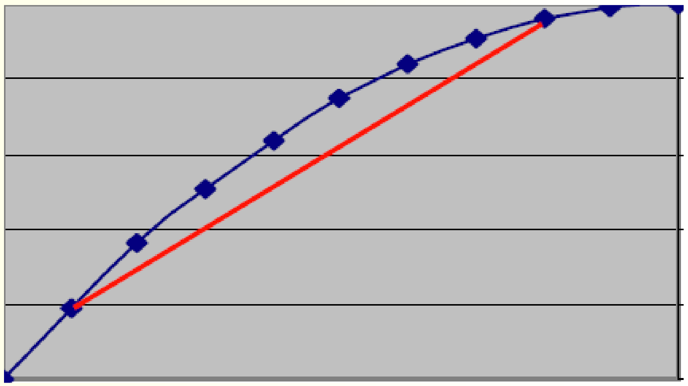
- A function f(x) is **convex** if linear interpolation between any two points x and y does not yield a lower value than the function



- Mathematically

$$f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$$
for all x, y and $\alpha \in [0,1]$

- A function f(x) is **concave** if linear interpolation between any two points x and y does not yield a higher value than the function

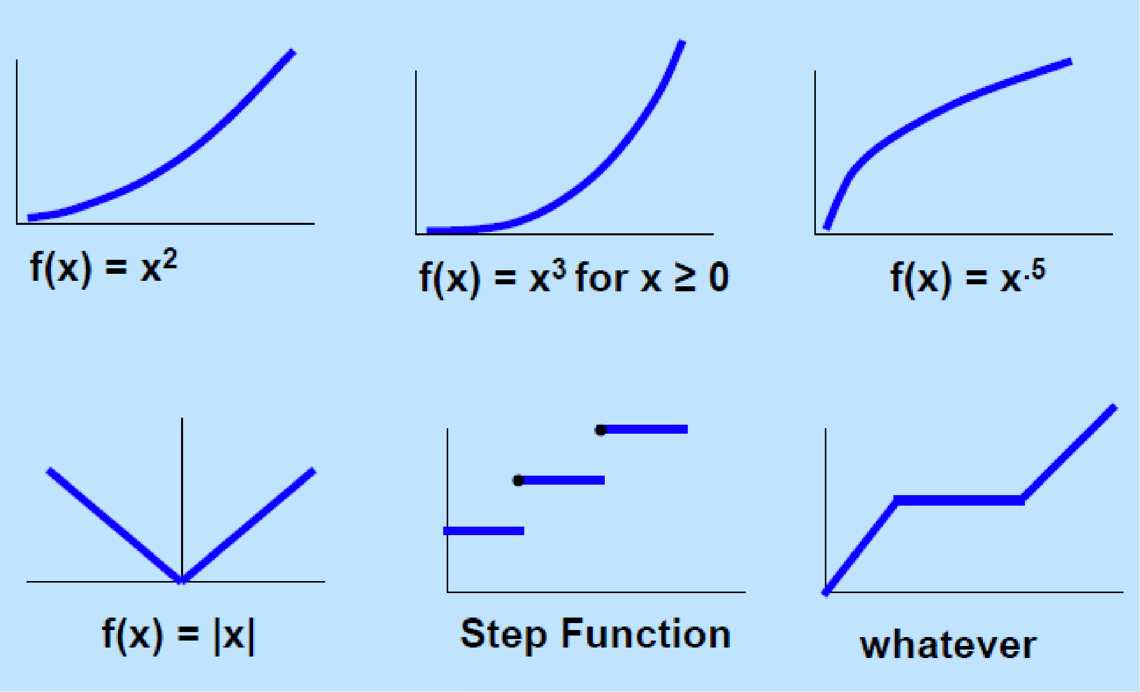


- Mathematically

$$f(\alpha x+(1-\alpha)y) \geq \alpha f(x)+(1-\alpha)f(y)$$
for all x, y and $\alpha \in [0,1]$

Convex and concave functions

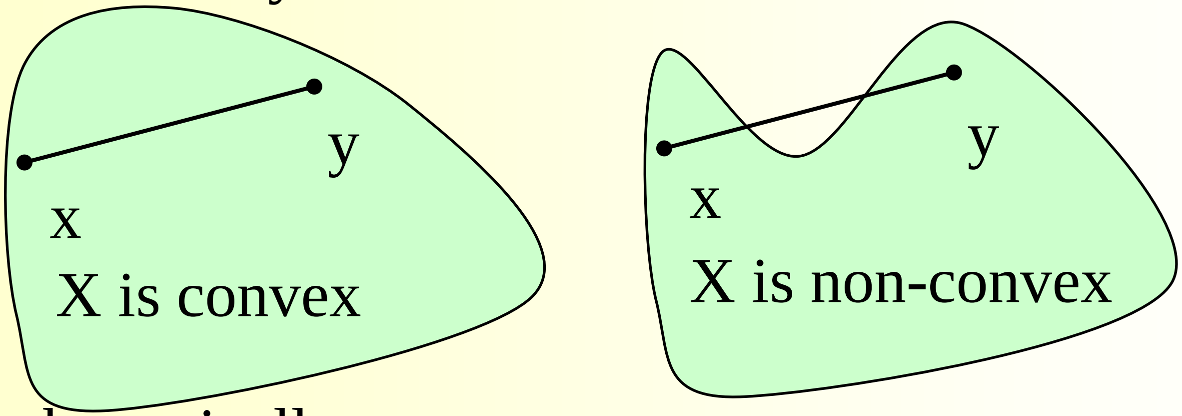
- Which functions are convex and which are concave?



  - Some functions are neither convex nor concave
  - If f(x) is convex, then –f(x) is concave and vice versa
  - **Only linear functions are both convex and concave**

Convex set

- A set X is **convex** if the line segment connecting any two points x and y of the set is in the set



- Mathematically
  - If $\mathbf{x},\mathbf{y} \in X$, then $\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in X$ for all $\alpha \in [0,1]$
- A constraint $g(\mathbf{x}) \leq 0$ defines a convex set if $g(\mathbf{x})$ is a convex function.
- The intersection of convex sets is a convex set
  - Thus multiple constraints $g_i(\mathbf{x}) \leq 0$ with convex functions $g_i(\mathbf{x})$ define a convex set

Convex optimization problems

- Convex optimization problems are relatively easy to solve because
  - A local optimum is also a global optimum
  - They can be solved using hill-climbing strategy: starting from any feasible point move in a direction where f(x) improves while maintaining feasibility
  - If the functions f(), **g**(), **h**() are smooth (first derivatives are continuous), various gradient-based methods can be used to identify improving directions
- Non-convex problems are difficult, because a local optimum is not in the general case a global optimum

Linear programming (LP) models

- An LP model has a linear objective function $f(\mathbf{x})$ and linear constraints $g_i(\mathbf{x})$:

$$\min (\max)\ c_1x_1 + c_2\, x_2 + \dots\ c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \dots\ a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\ a_{2n}x_n \le b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \dots\ a_{mn}x_n \le b_m$$

- Typical matrix representation:

$$\min (\max)\ \mathbf{cx}$$

s.t.

$$A\mathbf{x} \le \mathbf{b}$$

$\mathbf{x} \ge 0$ // traditionally variables are non-negative

Linear programming (LP) models

- Special case of convex problems
  - f(**x**), **g**(**x**) and **h**(**x**) are linear functions of **x**
  - The constraints are (hyper-) planes in *n* dimensions
  - The feasible area is an *n*-dimensional polyhedron
  - The optimum is at a corner point at the intersection between some constraint planes
- Very efficient solution algorithms for LP models exist
  - The **Simplex algorithm** can solve LP models with millions of variables and constraints
- Non-linear convex problems can be approximated by LP models with arbitrarily good accuracy
- Non-convex problems cannot be represented as LP models

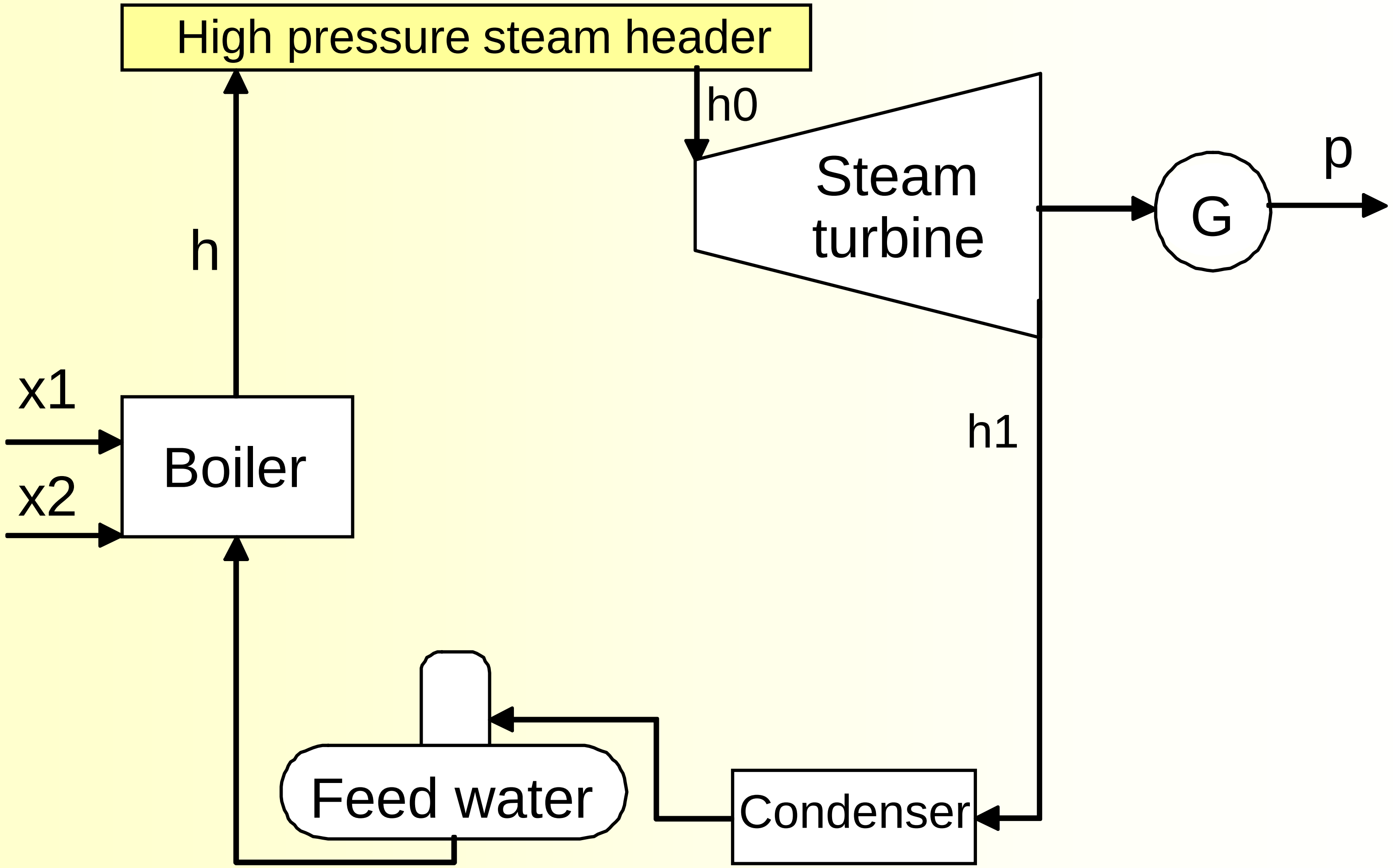
How to define an LP model?

1. Write down a verbal explanation of what is the goal or purpose of the model
   - E.g. to minimize costs or maximize profit of some specific operation or activity
2. Define the **decision variables** (and parameters)
   - Use as descriptive or generic names as you like: x1, x2, fuel, …
   - Give short description for them
   - Also specify the unit (MWh, GJ, €/kg, m3/s, …)
3. Define the **objective function** to minimize or maximize as a *linear function* of the variables
4. Define the **constraints** as *linear* inequality or equality constraints for the variables

LP example: Dual fuel condensing power plant

- Boiler can use two different fuels simultaneously in any proportion
- Boiler produces high pressure steam for a turbine driving a generator to produce electricity
- After turbine, steam is condensed back into water
- Fuels have different prices and consumption ratios
- Produced power is sold to market
- Typical objective is to **maximize profit** = revenue from selling power minus fuel costs

Dual fuel condensing power plant



LP example: Dual fuel condensing power plant

- Maximize profit during each hour of operation
- Decision variables
  - x1, x2 fuel consumption (MWh)
  - p power output (MWh)
- Parameters
  - r1, r2 consumption ratios for fuels (1)
  - c1, c2, c price for fuels and power (€/MWh)
  - $x1^{max}$, $x2^{max}$ upper bounds for fuel consumption (MWh)
  - b hourly maximal production capacity (MWh)

LP example: Dual fuel condensing power plant

- Objective function

  max c*p - c1*x1 - c2*x2 // power sales minus fuel costs

- Constraints

  p = x1/r1 + x2/r2 // power depends on fuel use

  $p \le b$ // capacity limit

  $x1 \le x1^{max}$, $x2 \le x2^{max}$, $x1, x2 \ge 0$

- Substitute expression for p to eliminate third variable

  max (c/r1-c1)*x1 + (c/r2 - c2)*x2

  $x1/r1 + x2/r2 \le b$ // capacity limit

  $x1 \le x1^{max}$, $x2 \le x2^{max}$, $x1, x2 \ge 0$

LP example: Dual fuel condensing power plant, numerical

- Parameters
  - Fuel consumption ratios (r1, r2) = (3.33, 2.5)
  - Fuel & power prices (c1, c2, c) = (20, 25, 80) €/MWh
  - Upper bounds for fuels ($x1^{max}$, $x2^{max}$) = (150,100) MWh
  - Production capacity b = 60 MWh

max (80/3.33-20)*x1 + (80/2.5-25)*x2 = 4*x1 + 7*x2

$0.3 \cdot x1 + 0.4 \cdot x2 \leq 60$

$x1 \leq 150$

$x2 \leq 100$

$x1, x2 \geq 0$

Graphical representation of LP models

- Models with two variables can be represented and solved graphically
  - Linear constraints are drawn as lines
    - The feasible region appears as a polygon
    - The feasible region may be unbounded in some direction
    - If the constraints are contradictory, the feasible region is empty and the model is infeasible
  - Level curves of objective function f(x) = K = constant are draw as dotted lines
    - Optimum is where a level curve touches the feasible region with with maximal or minimal K
    - This happens at some corner
    - If two corners yield optimal value, all points on the connecting edge are optimal (infinite number of optima)

LP example:
Power plant model, graphical representation

max 4*x1 + 7*x2

0.3*x1 + 0.4*x2 ≤ 60

x1 ≤ 150

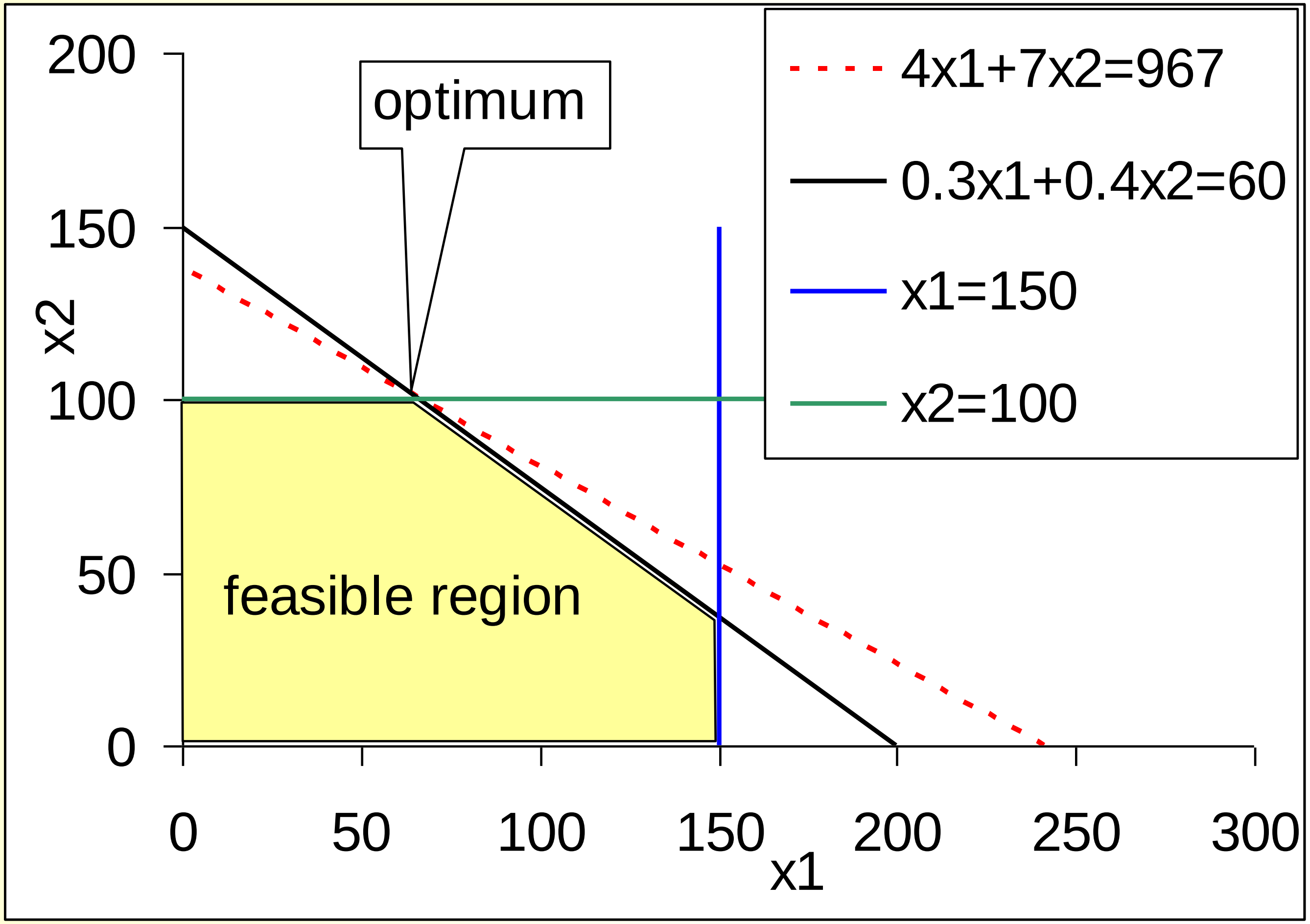
x2 ≤ 100

x1, x2 ≥ 0

Optimum at

x2=100

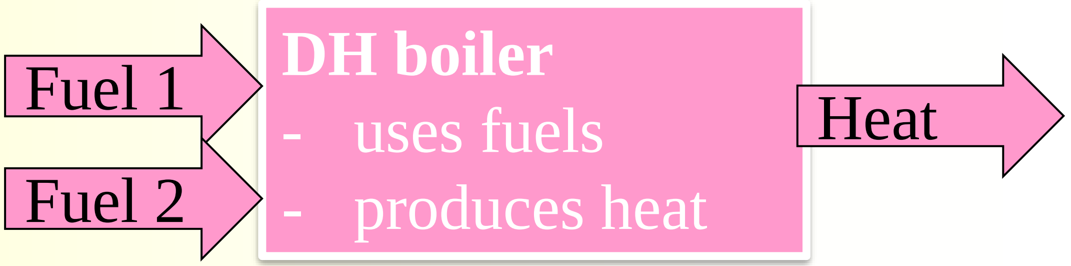
x1=66.7



Properties of LP models

- Similar to a general optimization problem, an LP problem can be
  - **Feasible**, if one or more feasible solutions exist
  - **Infeasible**, if no feasible solutions exist, i.e. constraints are conflicting
    - Example: min 0 s.t. $x \geq 1$, $x \leq 0$
  - **Unbounded**, if infinitely good solutions exist
    - Example: max x s.t. $x \geq 0$
- An LP problem has infinite number of optima if two or more corner solutions yield optimal value
  - Then all convex combinations of optimal corner solutions are optimal

LP example: DH boiler



- A dual fuel boiler to produce district heat
  - Goal to meet demand (MWh) as cheaply as possible
  - Decision variables
    - x1, x2 fuel consumption (MWh)
  - Parameters
    - r1, r2 consumption ratios for fuels (1)
    - c1, c2 prices for fuels (€/MWh)
    - $x1^{max}$, $x2^{max}$ upper bounds for fuel consumption (MWh)
    - b demand of heat

$\min c1 * x1 + c2 * x2$

$x1/r1 + x2/r2 \geq b$ // allowed to produce excess

$x1 \leq x1^{max}$, $x2 \leq x2^{max}$, $x1, x2 \geq 0$

LP example: DH boiler, numerical example

- Parameters
  - Fuel consumption ratios (r1, r2) = (1.25, 1.11)
  - Fuel prices (c1, c2) = (20, 25) €/MWh
  - Upper bounds for fuels ($x1^{max}$, $x2^{max}$) = (150,100) MWh
  - Heat demand b = 120

min 20*x1 + 25*x2;

0.8*x1 + 0.9*x2 ≥ 120;

x1 ≤ 150;

x2 ≤ 100;

x1, x2 ≥ 0;

Review questions

- Please review lecture material and see that you can answer the review
  1. Is the optimization problem max $x^2 + y^2$ s.t. $x,y \geq 0$ feasible, infeasible or unbounded? Why?
  2. Give a feasible solution to the above problem.
  3. How many optimal solutions does the problem max x^2 s.t. $-5 \leq x \leq 5$ have?
  4. Transform max x s.t. $x \leq 5$ replacing inequality constraint by equality constraint.
  5. Transform max x s.t. $x \leq 5$ into an unconstrained optimization problem.
  6. Why is classification of optimization problems important?
  7. Classify the following optimization problem: min $x^2 + y^2$ s.t. $x,y \geq 0$, $x \in R$, $y \in N$
  8. Why is LP modelling so common?
  9. Why are convex optimization problems relatively easy to solve?
  10. Give an example of an optimization problem which is difficult or impossible to solve.
  11. Does LP apply to non-linear problems? Why, or why not?
  12. When can an LP problem have infinite number of optimal solutions?
  13. Give an example of an infeasible LP problem
  14. Give an examle of a feasible LP problem without optimal solution
  15. Give an example of an LP problem with infinite number of optima