Short introduction to finite element method

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Outline

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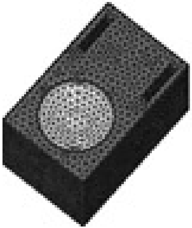
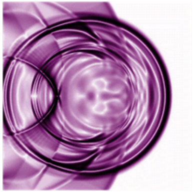
Introduction

- **Finite Element Method (FEM):** A numerical technique for finding approximate solutions to boundary value problems for partial differential equations.
- Originated in the 1940s and 1950s for structural engineering applications.
- Some references
  - ”The Finite Element Method: Its Basis and Fundamentals” by O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu
  - ”A First Course in the Finite Element Method” by D.L. Logan
  - ”The Mathematical Theory of Finite Element Methods” by S.C. Brenner and L.R. Scott





Key components

Key Concepts

- **Element:** A small, simple shape used to approximate the behavior of a larger, more complex structure.
- **Mesh:** Division of the structure into interconnected elements.

Workflow

- **Discretization:** Divide the physical space into elements.
- **Interpolation:** Define behavior within each element.
- **Assembly:** Combine element equations to form the system equations.
- **Solution:** Solve the system equations for the unknowns.

Pros and cons

Advantages

- **Versatile:** Applicable to a wide range of problems.
- **Efficiency:** Efficient for complex geometries.

Challenges

- **Mesh Generation:** Creating a suitable mesh can be challenging.
- **Validation:** Ensuring the accuracy of results.

Commercial software tools include Abaqus, ANSYS, and Comsol Multiphysics.

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Idea of Galerkin FEM

Let us consider a simple 2nd order ordinary differential equation

$$-\frac{d}{dx}\left(\frac{du(x)}{dx}\right) + q(x)u(x) = f(x), \quad a < x < b$$

With boundary conditions:

$$u(a) = 0$$
$$u(b) = 0$$

Let us assume that the solution $u \in X$ exists (X will be defined later)

Idea of Galerkin FEM (Contd.)

Let X_h be a finite dimensional subspace of X and let $u_h \in X_h$ be an approximation for u

$$u(x) \approx u_h = \sum_{\ell=1}^{N} \alpha_\ell v_\ell(x),$$

where α_ℓ are unknown coefficients and $v_\ell(x) \in X_h$.

Questions arise:

1. How to choose v_ℓ?
2. How to determine α_ℓ?

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Variational Formulation

- Key concept: Minimization of a functional.
- Derivation of weak form from the strong form of the problem.

Variational Formulation

- Key concept: Minimization of a functional.
- Derivation of weak form from the strong form of the problem.

Let us continue with the previously studied 2nd order system

- Multiply the system by a test function $v = v(x)$ and integrate over the domain:

$$\int_a^b \left(-\frac{d}{dx}\left(\frac{du}{dx}\right) + qu - f \right) v \, dx = 0, \quad \forall v \in X$$

Variational Formulation for a 1D Problem

We can re-write the system as

$$\int_a^b \left(-\frac{d}{dx}\left(\frac{du}{dx}\right)\right) v \, dx + \int_a^b quv \, dx - \int_a^b fv \, dx = 0, \quad \forall v \in X$$

Apply integration by parts:

$$\int_a^b \frac{du}{dx}\frac{dv}{dx} \, dx - \left(\frac{du}{dx}v\right)\Big|_a^b + \int_a^b quv \, dx - \int_a^b fv \, dx = 0, \quad \forall v \in X$$

Boundary conditions given above leads $v(a) = v(b) = 0$ (v is in the same space as u).

Variational Formulation for a 1D Problem (Contd.)

Apply boundary conditions and obtain the weak form:

$$\begin{aligned}\int_a^b \frac{du}{dx}\frac{dv}{dx}\,dx + \int_a^b quv\,dx &= \int_a^b fv\,dx, \quad \forall v \in X \\ a(u,v) &= F(v), \quad \forall v \in X\end{aligned}$$

- This is the variational form and it is equivalent with the original PDE.
- Solving this solution (in weak sense) for the original problem.
- $a(u, v)$ is so-called bilinear form.

Outline

Discretization

- Variational form can be discretised using Galerkin finite element approximation:
- We set

$$u \approx u_h = \sum_{\ell=1}^{N} \alpha_\ell \phi_\ell \quad \text{and choose} \quad v = \phi_j$$

Now we get

$$\begin{aligned} a(u_h, \phi_j) &= F(\phi_j), \quad \forall j = 1, \ldots, N \\ \int_a^b \left(\sum_{\ell=1}^{N} \alpha_\ell \frac{d\phi_\ell}{dx} \frac{d\phi_j}{dx} \right) dx &+ \\ \int_a^b q \left(\sum_{\ell=1}^{N} \alpha_\ell \phi_\ell \phi_j \right) dx &= \int_a^b f \phi_j \, dx, \quad \forall j = 1, \ldots, N \end{aligned}$$

Discretization (Contd.)

- The above system can be written as $K\alpha = b$, where

$$\begin{aligned} k_{j\ell} &= K(j,\ell) = \int_a^b \left(\frac{d\phi_\ell}{dx}\frac{d\phi_j}{dx} + q\phi_\ell\phi_j \right) dx \\ b_j &= b(j) = \int_a^b f\phi_j \, dx \\ \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_N)^\top \end{aligned}$$

- From this, the unknowns α can be solved (formally) as $\alpha = K^{-1}b$ and then

$$u_h = \sum_{\ell=1}^{N} \alpha_\ell \phi_\ell.$$

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1D numerical example

Let us consider a 2nd order ordinary differential equation

$$-u'' + u = -2x^2 + 4x, \quad x \in \Omega = [0, 1]$$

With boundary conditions:

$$u'(0) = 4$$
$$u'(1) = 0$$

Note that $'$ denotes derivative with respect to x.

1D numerical example (Contd.)

- First step would be to derive the weak for form the system
- As studied in the previous slides, we multiply the system equation with test function v and integrate over the domain G

$$-\int_0^1 u''v\,dx + \int_0^1 uv\,dx = \int_0^1 (-2x^2 + 4x)v\,dx, \quad \forall v \in G$$

Apply integration by parts for the first term:

$$\int_0^1 u'v'\,dx - \underbrace{(u'v)\Big|_0^1}_{=-4v(0)} + \int_0^1 uv\,dx = \int_0^1 (-2x^2 + 4x)v\,dx, \quad \forall v \in G$$

1D numerical example (Contd.)

- As earlier, we set

$$u \approx u_h = \sum_{\ell=1}^{N} \alpha_\ell \phi_\ell \quad \text{and choose} \quad v = \phi_j$$

Now we get

$$\begin{aligned} \int_0^1 \left(\sum_{\ell=1}^{N} \alpha_\ell \phi_\ell' \phi_j' \right) dx \quad &+ \quad 4\phi_j(0) + \\ \int_0^1 \left(\sum_{\ell=1}^{N} \alpha_\ell \phi_\ell \phi_j \right) dx \quad &= \quad \int_0^1 f\phi_j \, dx, \quad \forall j = 1, \dots, N \end{aligned}$$

where $f = -2x^2 + 4x$

1D numerical example (Contd.)

- Reordering leads

$$\int_0^1 \left(\sum_{\ell=1}^{N} \alpha_\ell \phi_\ell' \phi_j' + \sum_{\ell=1}^{N} \alpha_\ell \phi_\ell \phi_j \right) \, dx \quad = \quad \int_0^1 f \phi_j \, dx - 4\phi_j(0),$$

that can be written as $(S + M)\alpha = K\alpha = b$, where

$$\begin{aligned} k_{j\ell} &= K(j, \ell) = \int_0^1 \left(\phi_\ell' \phi_j' + \phi_\ell \phi_j \right) \, dx \\ b_j &= b(j) = \int_0^1 f \phi_j \, dx - 4\phi_j(0) \\ \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_N)^\top \end{aligned}$$

- From this, the unknowns α can be solved as $\alpha = K^{-1} b$

1D numerical example (Contd.)

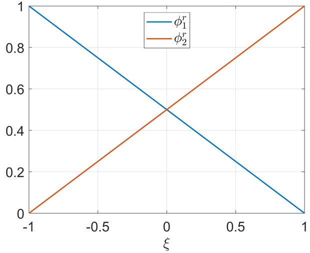
- Integrals can be computed in the reference element (crucial step to make the solver faster)
- We will skip the details but the idea relies to integral of composite functions (familiar from integral calculus)
- For general 3D case it reads

$$\int_{G_e} g(x,y,z)\,dxdydz = \int_{G^0} (g \circ F^e)(\xi,\eta,\gamma)|J_{F^e}|\,d\xi d\eta d\gamma,$$

where $|J_{F^e}|$ is the determinant of the Jacobian related to the mapping F^e

1D numerical example (Contd.)

- Let us assume linear basis functions, those are for the reference element $\Omega_r \in [-1, 1]$ as



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\begin{aligned}
\phi_1^r(\xi) &= (1-\xi)/2 \\
\phi_2^r(\xi) &= (1+\xi)/2
\end{aligned}
$$

- The global coordinate x within an element is related to the local coordinate ξ by (i.e. the mapping F^e):

$$x(\xi) = h/2(1+\xi) + x_i,$$

  where h is the length of the element and x_i is the starting point of the element.

1D numerical example (Contd.)

- How to build matrix K
- As an example (one term for the local K matrix):

$$
\begin{aligned}
K^r(1,1) &= \ldots \\
&= \int_{-1}^{1} \left(\frac{\phi_1^{r\prime}(\xi)}{F^{e\prime}(\xi)} \frac{\phi_1^{r\prime}(\xi)}{F^{e\prime}(\xi)} + \phi_1^r(\xi)\phi_1^r(\xi) \right) |dF^e/d\xi|\, d\xi \\
&= \int_{-1}^{1} \left(1/h^2 + \phi_1(\xi)\phi_1(\xi)\right) h/2\, d\xi = 1/h + h/3
\end{aligned}
$$

- Note that for the first term, we applied the chain rule of differentiation:

$$(g \circ F^e)'(\xi) = g'(F^e(\xi))F^{e\prime}(\xi)$$

1D numerical example (Contd.)

- How to build right hand side b
- We express the term f using the linear basis functions as $f_h = \sum_{\ell=1}^{N} f_\ell \phi_\ell$

$$\begin{aligned} b(j) &= \int_0^1 \sum_{\ell=1}^{N} \left(f_\ell \phi_\ell \phi_j \right) \, dx - 4\phi_j(0) \\ &= \int_{-1}^1 \sum_{\ell=1}^{N} \left(f_\ell \phi_\ell^r \phi_j^r \right) \, d\xi - 4\phi_j(0) \end{aligned}$$

- As seen, this reduces to the same integral as studied previously for the matrix K and hence we can use $b = Mf^\top$

1D numerical example (Contd.)

- Analytic solution for the studied problem is

$$u_{exact} = -2x^2 + 4x - 4$$

- Let us examine the effect of the grid on the numerical accuracy (in Matlab)
- Convergence order:

$$O = \frac{\ln\left(e_{\ell+1}/e_{\ell}\right)}{\ln\left(h_{\ell+1}/h_{\ell}\right)},$$

where e denotes the L_2 error for different grids ℓ

Outline

Motivation for Numerical Integration

- In FEM, integrals over elements are a crucial part of the formulation.
- Analytical integration may not be feasible for complex geometries and material properties.
- Numerical integration provides an efficient approach to approximate these integrals.

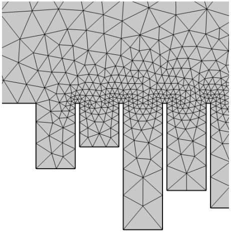
Gaussian Quadrature - A widely used approach:

- Based on the idea of approximating the integral using weighted sum at specific points.
- Nodes and weights are pre-determined for different orders of quadrature.

Outline

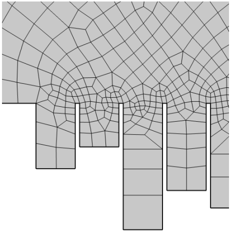
Introduction to 2D Problems

- In 2D, physical domains are represented by surfaces.
- Nodes and elements are extended into two dimensions.
- Nodal points now have two coordinates (x, y).



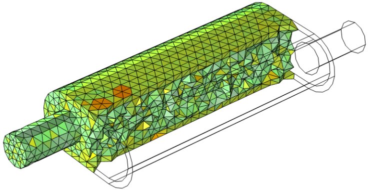
Mesh generation in 2D:

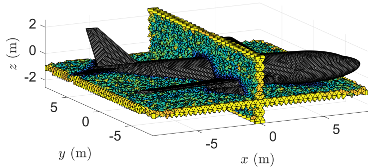
- Triangular or quadrilateral elements are commonly used.
- Mesh generation becomes more complex than in 1D.



Extension to 3D Problems

- In 3D, physical domains are represented in three-dimensional space.
- Nodes now have three coordinates (x, y, z).
- Tetrahedral or hexahedral elements are commonly used in 3D.
- Mesh generation becomes more challenging but follows the same principles as in 2D.





Challenges and Considerations

- Increased computational complexity: More nodes, more elements, and more degrees of freedom.
- Choice of element type: Triangular, quadrilateral, tetrahedral, hexahedral, etc.
- Mesh quality considerations: Delaunay conditions in 2D, aspect ratios in 3D.
- Visualization challenges: Representing 3D structures in a comprehensible manner.