Aalto University
School of Electrical Engineering

Vibrations of strings and membranes

ELEC-E5610 Acoustics and the Physics of Sound, Lecture 2

Georg Götz

Department of Information and Communications Engineering
Aalto University School of Electrical Engineering

October 26, 2023

1

Ideal strings

1 The ideal string

An **ideal string** is a fictitious entity with certain special properties. Namely, it is

- homogeneous
- perfectly flexible
- lossless

1 The ideal string

An **ideal string** is a fictitious entity with certain special properties. Namely, it is

- homogeneous
- perfectly flexible
- lossless

It has three quantities that govern its behavior:

- linear density μ [$\frac{\text{kg}}{\text{m}}$]
- tension T_0 [N]
- length L [m]

In the following, we will consider the movement of string in one plane only. The longitudinal coordinate is x and the transversal displacement is denoted with y.

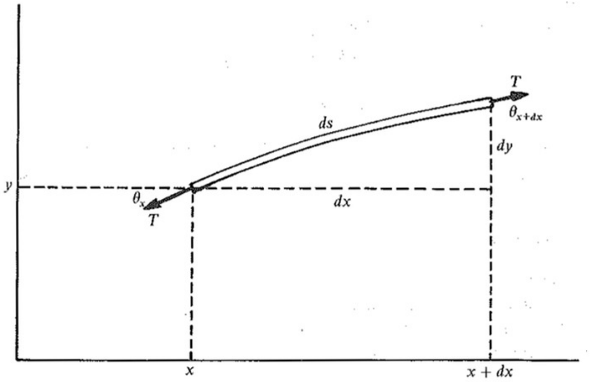


Fig. 2.3. Segments of a string with tension T.

The two opposing forces here are

- inertia (mass × acceleration)
- spring force (tension × curvature)

If the string displacement is moderate, i.e. $\frac{\partial y}{\partial x} \ll 1$, the movement of the ideal string can be characterized with the 1D wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

where $c = \sqrt{\frac{T_0}{\mu}}$.

1 The Wave Equation

The two opposing forces here are

- inertia (mass $\times$ acceleration)
- spring force (tension $\times$ curvature)

If the string displacement is moderate, i.e. $\frac{\partial y}{\partial x} \ll 1$, the movement of the ideal string can be characterized with the 1D wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

where $c = \sqrt{\frac{T_0}{\mu}}$.

1 The Wave Equation

The two opposing forces here are

- inertia (mass $\times$ acceleration)
- spring force (tension $\times$ curvature)

If the string displacement is moderate, i.e. $\frac{\partial y}{\partial x} \ll 1$, the movement of the ideal string can be characterized with the 1D wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

where $c = \sqrt{\frac{T_0}{\mu}}$.

1 Two Solutions

Two main solutions for Eq. (1) are

d'Alembert's solution where the string vibration is seen as two waves traveling in opposite directions

$$y(x, t) = g_1(ct - x) + g_2(ct + x) \tag{2}$$

1 Two Solutions

Two main solutions for Eq. (1) are

d'Alembert's solution where the string vibration is seen as two waves traveling in opposite directions

$$y(x, t) = g_1(ct - x) + g_2(ct + x) \tag{2}$$

(animations: `https://www.acs.psu.edu/drussell/Demos/Pluck-Fourier/Pluck-Fourier.html`

`http://www.phys.unsw.edu.au/jw/strings.html`)

1 Two Solutions

Two main solutions for Eq. (1) are

d'Alembert's solution where the string vibration is seen as two waves traveling in opposite directions

$$y(x,t) = g_1(ct - x) + g_2(ct + x) \tag{2}$$

(animations: `https://www.acs.psu.edu/drussell/Demos/Pluck-Fourier/Pluck-Fourier.html`

`http://www.phys.unsw.edu.au/jw/strings.html`)

Bernoulli's solution where the vibration is seen as a superposition of standing wave modes

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

(R&F:(2.13))

1 Bernoulli's Solution

Let's take a closer look on Eq. (R&F:(2.13)):

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

1 Bernoulli's Solution

Let's take a closer look on Eq. (R&F:(2.13)):

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

String vibration is

1 Bernoulli's Solution

Let's take a closer look on Eq. (R&F:(2.13)):

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

String vibration is

- **a sum over mode number n of...**

1 Bernoulli's Solution

Let's take a closer look on Eq. (R&F:(2.13)):

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

String vibration is

- a sum over mode number n of...
- **spatial sinusoidal terms (modes), multiplied by...**

1 Bernoulli's Solution

Let's take a closer look on Eq. (R&F:(2.13)):

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

String vibration is

- a sum over mode number n of...
- spatial sinusoidal terms (modes), multiplied by...
- **temporal sinusoidal terms (vibration in time)**

Let's take a closer look on Eq. (R&F:(2.13)):

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

String vibration is

- a sum over mode number n of...
- spatial sinusoidal terms (modes), multiplied by...
- temporal sinusoidal terms (vibration in time)
  - A_n and B_n together define the amplitude of each frequency component

1 Bernoulli's Solution II

In addition to the physical string parameters L and c, the spectrum of the vibration is defined by A_n and B_n.

1 Bernoulli's Solution II

In addition to the physical string parameters L and c, the spectrum of the vibration is defined by A_n and B_n. How are A_n and B_n defined?

1 Bernoulli's Solution II

In addition to the physical string parameters L and c, the spectrum of the vibration is defined by A_n and B_n. How are A_n and B_n defined?
- By excitation, i. e. the initial conditions for the velocity and displacement:

$$A_n = \frac{2}{\omega_n L} \int_0^L \dot{y}(x,0) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{(R\&F:(2.17))}$$

$$B_n = \frac{2}{L} \int_0^L y(x,0) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{(R\&F:(2.18))}$$

1 Bernoulli's Solution II

In addition to the physical string parameters L and c, the spectrum of the vibration is defined by A_n and B_n. How are A_n and B_n defined?
- By excitation, i. e. the initial conditions for the velocity and displacement:

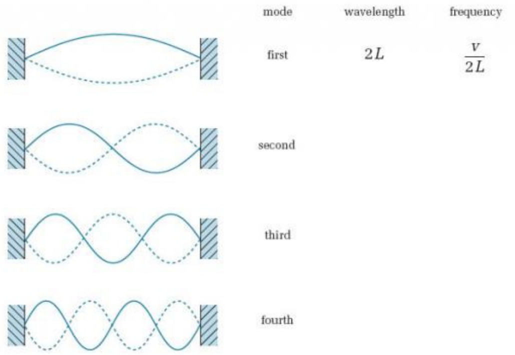
$$A_n = \frac{2}{\omega_n L} \int_0^L \dot{y}(x,0) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{(R\&F:(2.17))}$$

$$B_n = \frac{2}{L} \int_0^L y(x,0) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{(R\&F:(2.18))}$$

For plucked string $\dot{y}(x,0) = 0 \Rightarrow A_n = 0$,
for a struck string $y(x,0) = 0 \Rightarrow B_n = 0$.

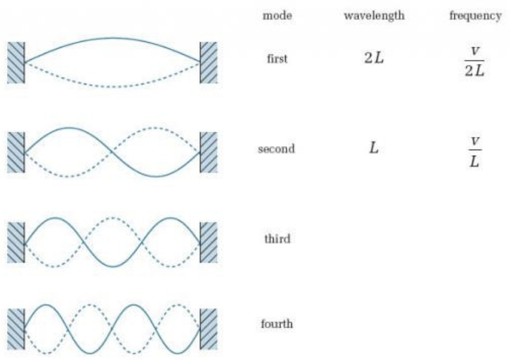
1 Bernoulli's Solution III

Recall that the spatial term in Eq. (R&F:(2.13)):
$y_n = \sin\left(\frac{n\pi x}{L}\right)$ corresponds to different modes...



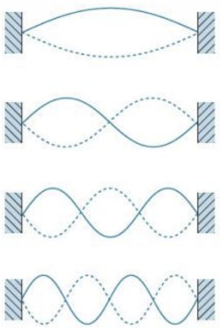
1 Bernoulli's Solution III

Recall that the spatial term in Eq. (R&F:(2.13)):
$y_n = \sin\left(\frac{n\pi x}{L}\right)$ corresponds to different modes...



1 Bernoulli's Solution III

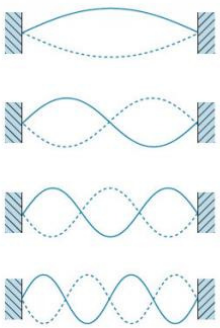
Recall that the spatial term in Eq. (R&F:(2.13)):
$y_n = \sin\left(\frac{n\pi x}{L}\right)$ corresponds to different modes...



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth		

1 Bernoulli's Solution III

Recall that the spatial term in Eq. (R&F:(2.13)):
$y_n = \sin\left(\frac{n\pi x}{L}\right)$ corresponds to different modes...



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

1 Bernoulli's Solution IV

...furthermore, the temporal term

$y_t = \left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$ vibrates at frequencies

$$f_n = n\frac{c}{2L} = nf_0 \quad (3)$$

1 Bernoulli's Solution IV

...furthermore, the temporal term

$y_t = \left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$ vibrates at frequencies

$$f_n = n\frac{c}{2L} = nf_0 \tag{3}$$

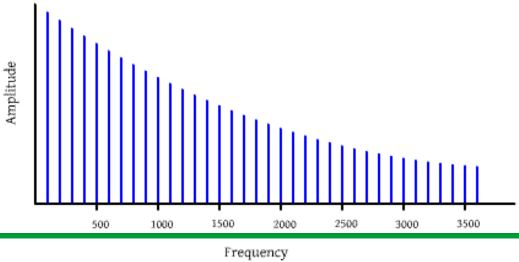
Vibration components evenly spaced in frequency!

1 Bernoulli's Solution IV

...furthermore, the temporal term $y_t = \left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$ vibrates at frequencies

$$f_n = n\frac{c}{2L} = nf_0 \tag{3}$$

Vibration components evenly spaced in frequency! ⇒ harmonic spectrum!



1 Free Vibration

Free vibration means that the string is excited with some initial conditions and then left to vibrate on its own (https://www.youtube.com/watch?v=_X72on6CSL0).

1 Free Vibration

Free vibration means that the string is excited with some initial conditions and then left to vibrate on its own (https://www.youtube.com/watch?v=_X72on6CSL0). Let's consider striking a string at some location, and see how Bernoulli's solution is affected.

1 Free Vibration

Free vibration means that the string is excited with some initial conditions and then left to vibrate on its own (https://www.youtube.com/watch?v=_X72on6CSL0). Let's consider striking a string at some location, and see how Bernoulli's solution is affected.

- add an excitation force to the wave equation: $\ddot{y} - c^2 y'' = f(x,t)$ [Note that $y'' = \frac{\partial^2 y}{\partial x^2}$]
- consider the force as an impulse at some location x_0: $f(x,t) = \delta(x - x_0)\delta(t)$, where δ is Dirac's delta function

1 Free Vibration

Free vibration means that the string is excited with some initial conditions and then left to vibrate on its own (https://www.youtube.com/watch?v=_X72on6CSL0). Let's consider striking a string at some location, and see how Bernoulli's solution is affected.

- add an excitation force to the wave equation: $\ddot{y} - c^2 y'' = f(x, t)$ [Note that $y'' = \frac{\partial^2 y}{\partial x^2}$]
- consider the force as an impulse at some location x_0: $f(x, t) = \delta(x - x_0)\delta(t)$, where δ is Dirac's delta function
- However, instead of force, we would need an initial velocity or displacement to calculate A_n and/or B_n. How to proceed?

1 Free Vibration II

- So, our force impulse becomes initial acceleration $\ddot{y}(x,0) = \frac{\delta(x-x_0)\delta(t)}{\mu}$

1 Free Vibration II

- So, our force impulse becomes initial acceleration $\ddot{y}(x, 0) = \frac{\delta(x-x_0)\delta(t)}{\mu}$
- initial velocity becomes $\dot{y}(x, 0) = \int \ddot{y}(x, 0)dt = \frac{\delta(x-x_0)}{\mu}$

1 Free Vibration II

- So, our force impulse becomes initial acceleration $\ddot{y}(x,0) = \frac{\delta(x-x_0)\delta(t)}{\mu}$
- initial velocity becomes $\dot{y}(x,0) = \int \ddot{y}(x,0)dt = \frac{\delta(x-x_0)}{\mu}$
- while the initial displacement is $y(x,0) = 0$

1 Free Vibration II

- So, our force impulse becomes initial acceleration $\ddot{y}(x,0) = \frac{\delta(x-x_0)\delta(t)}{\mu}$
- initial velocity becomes $\dot{y}(x,0) = \int \ddot{y}(x,0)dt = \frac{\delta(x-x_0)}{\mu}$
- while the initial displacement is $y(x,0) = 0$

Insert the initial conditions into Eqs. R&F:(2.17) and R&F:(2.18):

$$A_n = \frac{2}{n\pi c} \int_0^L \dot{y}(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{L} \int_0^L y(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$$

1 Free Vibration II

- So, our force impulse becomes initial acceleration $\ddot{y}(x,0) = \frac{\delta(x-x_0)\delta(t)}{\mu}$
- initial velocity becomes $\dot{y}(x,0) = \int \ddot{y}(x,0)dt = \frac{\delta(x-x_0)}{\mu}$
- while the initial displacement is $y(x,0) = 0$

Insert the initial conditions into Eqs. R&F:(2.17) and R&F:(2.18):

$$A_n = \frac{2}{n\pi c}\int_0^L \frac{\delta(x-x_0)}{\mu} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = 0$$

1 Free Vibration III

Next, evaluate the integral:

$$\int_0^L \frac{\delta(x - x_0)}{\mu} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{\mu} \sin\left(\frac{n\pi x_0}{L}\right)$$

1 Free Vibration III

Next, evaluate the integral:

$$\int_0^L \frac{\delta(x - x_0)}{\mu} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{\mu} \sin\left(\frac{n\pi x_0}{L}\right)$$

(when $x_0 \in [0, L]$).

1 Free Vibration III

Next, evaluate the integral:

$$\int_0^L \frac{\delta(x - x_0)}{\mu} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{\mu} \sin\left(\frac{n\pi x_0}{L}\right)$$

(when $x_0 \in [0, L]$). Thus, the string vibration is

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

where

$$A_n = \frac{2}{n\pi c\mu} \sin\left(\frac{n\pi x_0}{L}\right), \; B_n = 0$$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L}$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{2}{n\pi\mu c}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\right] \tag{4}$$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{2n}{n\pi\mu 2Lf_n}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi 2Lf_n t}{Ln}\right)\right] \tag{4}$$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{1}{\pi\mu L f_n}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\pi 2 f_n t\right)\right] \tag{4}$$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (4)$$

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin(\pi 2 f_n t) \quad (4)$$

consists of

- **spatial terms (eigenmodes)**

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (4)$$

consists of

- spatial terms (eigenmodes)
- **temporal terms (eigenfrequencies)**

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (4)$$

consists of

- spatial terms (eigenmodes)
- temporal terms (eigenfrequencies)
- **frequency-dependent scaling**

1 Free Vibration IV

Remember Eq. (3): $f_n = \frac{nc}{2L} \Leftrightarrow c = \frac{2Lf_n}{n}$ insert into the vibration equation:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (4)$$

consists of

- spatial terms (eigenmodes)
- temporal terms (eigenfrequencies)
- frequency-dependent scaling

1 String excited at x_0

After all the math, we obtained a nice equation for the vibration of a string, struck at x_0:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (5)$$

1 String excited at x_0

After all the math, we obtained a nice equation for the vibration of a string, struck at x_0:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (5)$$

What if $x_0 = L/2$?

1 String excited at x_0

After all the math, we obtained a nice equation for the vibration of a string, struck at x_0:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (5)$$

What if $x_0 = L/2$? The middle sine term becomes $\sin\left(\frac{n\pi}{2}\right)$

1 String excited at x_0

After all the math, we obtained a nice equation for the vibration of a string, struck at x_0:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (5)$$

What if $x_0 = L/2$? The middle sine term becomes $\sin\left(\frac{n\pi}{2}\right) = 0,$ when $n = 2, 4, 6, 8, \ldots \Rightarrow$ even harmonics absent!

1 String excited at x_0

After all the math, we obtained a nice equation for the vibration of a string, struck at x_0:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (5)$$

What if $x_0 = L/2$? The middle sine term becomes $\sin\left(\frac{n\pi}{2}\right) = 0$, when $n = 2, 4, 6, 8, \ldots \Rightarrow$ even harmonics absent! Generally:

- **if the string is excited at $\frac{1}{m}$ of it length, every mth harmonic will be missing**

1 String excited at x_0

After all the math, we obtained a nice equation for the vibration of a string, struck at x_0:

$$y(x,t) = \frac{1}{\pi L \mu} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) \frac{1}{f_n} \sin\left(\pi 2 f_n t\right) \quad (5)$$

What if $x_0 = L/2$? The middle sine term becomes $\sin\left(\frac{n\pi}{2}\right) = 0$, when $n = 2, 4, 6, 8, \ldots \Rightarrow$ even harmonics absent! Generally:

- **if the string is excited at $\frac{1}{m}$ of it length, every mth harmonic will be missing**
- if excitation location is moved towards string's end, fewer and fewer harmonics missing $\Rightarrow$ sound gets brighter!

1 String excited at x_0 II

An alternative (graphical) way to express the same idea: the closer the excitation is to the antinode of an eigenmode, the better it excites the corresponding eigenfrequency.

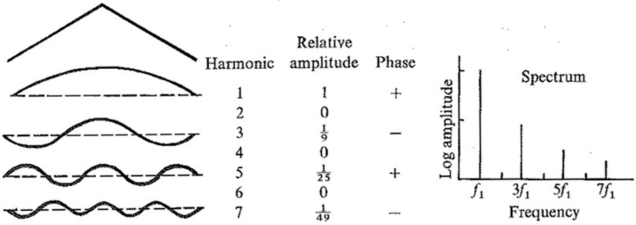


Fig. 2.5. Frequency analysis of a string plucked at its center. Odd-numbered modes of vibration add up in appropriate amplitude and phase to give the shape of the string.

1 String excited at x_0 III

Actually used in the piano! The piano hammer typically hits the string at $\frac{1}{7}$th of its length $\Rightarrow$ the 7th harmonic damped...

Table: Eigenfrequencies and closest notes for A_2 note.

n	f_n	note	f_{note}	error (Hz)
1	110	A_2	110	0
2	220	A_3	220	0
3	330	E_4	329.63	0.37
4	440	A_4	440	0
5	550	$C\#_5$	554.37	4.37
6	660	E_5	659.26	0.74
7	770	G_5	783.99	13.99
8	880	A_6	880	0

1 String excited at x_0 III

Actually used in the piano! The piano hammer typically hits the string at $\frac{1}{7}$th of its length $\Rightarrow$ the 7th harmonic damped...

Table: Eigenfrequencies and closest notes for A_2 note.

n	f_n	note	f_{note}	error (Hz)
1	110	A_2	110	0
2	220	A_3	220	0
3	330	E_4	329.63	0.37
4	440	A_4	440	0
5	550	$C\#_5$	554.37	4.37
6	660	E_5	659.26	0.74
7	770	G_5	783.99	**13.99**
8	880	A_6	880	0

...luckily, since it's so much out of tune!

When the excitation is continuous, the string vibration is considered *forced*.

- basically, the same mechanisms apply as what discussed above
- also, the frequency of the excitation force has an effect
  - the excitation must “match” both the spatial and temporal form of a mode, if that mode is to be excited
- a continuous excitation at an eigenfrequency exponentially increases the vibration amplitude
  - $\Rightarrow$ amplitude would become infinite, if it weren't for the losses

2

Losses, Stiffness, Nonlinearities, Other Polarizations

2 Loss Mechanisms

The most important loss mechanisms in a vibrating string are:

2 Loss Mechanisms

The most important loss mechanisms in a vibrating string are:

- damping caused by air viscosity

2 Loss Mechanisms

The most important loss mechanisms in a vibrating string are:

- damping caused by air viscosity
- internal losses

2 Loss Mechanisms

The most important loss mechanisms in a vibrating string are:

- damping caused by air viscosity
- internal losses
  - actually, a set of different thermo- and viscoelastic loss mechanisms

2 Loss Mechanisms

The most important loss mechanisms in a vibrating string are:

- damping caused by air viscosity
- internal losses
  - actually, a set of different thermo- and viscoelastic loss mechanisms
- transfer of mechanical energy through supports
  - depends on the connection impedance between the string and the body

2 Loss Mechanisms

The most important loss mechanisms in a vibrating string are:

- damping caused by air viscosity
- internal losses
  - actually, a set of different thermo- and viscoelastic loss mechanisms
- transfer of mechanical energy through supports
  - depends on the connection impedance between the string and the body

The combined effect of all losses may be expressed as a single force term $R(f)$

2 Effect of Losses

Strictly speaking, the loss term $R(f)$ depends not only on the frequency, but also on

- physical properties of the string
- string geometry
- properties of the air

2 Effect of Losses

Strictly speaking, the loss term $R(f)$ depends not only on the frequency, but also on

- physical properties of the string
- string geometry
- properties of the air

$\Rightarrow$ difficult to obtain theoretically! In practice, it is measured from the attenuation times at different frequencies.

Strictly speaking, the loss term $R(f)$ depends not only on the frequency, but also on

- physical properties of the string
- string geometry
- properties of the air

$\Rightarrow$ difficult to obtain theoretically! In practice, it is measured from the attenuation times at different frequencies. Results in an additional term to the wave equation:

$$\ddot{y} - c^2 y'' + 2R(f)\dot{y} = f(x,t) \qquad (6)$$

2 Effect of Losses

Strictly speaking, the loss term $R(f)$ depends not only on the frequency, but also on

- physical properties of the string
- string geometry
- properties of the air

$\Rightarrow$ difficult to obtain theoretically! In practice, it is measured from the attenuation times at different frequencies. Results in an additional term to the wave equation:

$$\ddot{y} - c^2 y'' + 2R(f)\dot{y} = f(x,t) \tag{6}$$

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0).

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0). The amount of the stiffness depends on

- cross-section area of the string A

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0). The amount of the stiffness depends on

- cross-section area of the string A
- linear density μ

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0). The amount of the stiffness depends on

- cross-section area of the string A
- linear density μ
- Young's modulus E (depends on the material)

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0). The amount of the stiffness depends on

- cross-section area of the string A
- linear density μ
- Young's modulus E (depends on the material)
- the radius of gyration κ (depends on string geometry)

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0). The amount of the stiffness depends on

- cross-section area of the string A
- linear density μ
- Young's modulus E (depends on the material)
- the radius of gyration κ (depends on string geometry)

Stiffness creates yet another force term to the wave equation:

$$\ddot{y} - c^2 y'' + 2R(f)\dot{y} + \frac{EA\kappa^2}{\mu} y'''' = f(x,t) \tag{7}$$

2 Effect of Stiffness

Real strings are never perfectly flexible, but have a nonzero stiffness. This internal stiffness generates another restoring force (in addition to the external tension T_0). The amount of the stiffness depends on

- cross-section area of the string A
- linear density μ
- Young's modulus E (depends on the material)
- the radius of gyration κ (depends on string geometry)

Stiffness creates yet another force term to the wave equation:

$$\ddot{y} - c^2 y'' + 2R(f)\dot{y} + \frac{EA\kappa^2}{\mu} y'''' = f(x,t) \quad (7)$$

2 Effect of Stiffness II

Stiffness causes the wave propagation velocity to become frequency-dependent

2 Effect of Stiffness II

Stiffness causes the wave propagation velocity to become frequency-dependent

- upper harmonics shift higher in frequency

2 Effect of Stiffness II

Stiffness causes the wave propagation velocity to become frequency-dependent

- upper harmonics shift higher in frequency
- $\Rightarrow$ the resulting tone no longer (strictly) harmonic!

2 Effect of Stiffness II

Stiffness causes the wave propagation velocity to become frequency-dependent

- upper harmonics shift higher in frequency
- $\Rightarrow$ the resulting tone no longer (strictly) harmonic!

Inharmonicity caused by stiffness has a significant impact on how pianos are tuned.

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization
  - termination impedance is different in different polarizations ⇒ different decay times ⇒ two-stage decay!

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization
  - termination impedance is different in different polarizations $\Rightarrow$ different decay times $\Rightarrow$ two-stage decay!
- longitudinal polarization

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization
  - termination impedance is different in different polarizations $\Rightarrow$ different decay times $\Rightarrow$ two-stage decay!
- longitudinal polarization
  - basically the same as transversal vibration, but the propagation velocity is different (see R&F: Sec. 2.14). Typically $c_L \gg c$

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization
  - termination impedance is different in different polarizations $\Rightarrow$ different decay times $\Rightarrow$ two-stage decay!
- longitudinal polarization
  - basically the same as transversal vibration, but the propagation velocity is different (see R&F: Sec. 2.14). Typically $c_L \gg c$
  - might connect to the instrument body (and become audible)

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization
  - termination impedance is different in different polarizations $\Rightarrow$ different decay times $\Rightarrow$ two-stage decay!
- longitudinal polarization
  - basically the same as transversal vibration, but the propagation velocity is different (see R&F: Sec. 2.14). Typically $c_L \gg c$
  - might connect to the instrument body (and become audible)
- rotational polarization
  - usually negligible, except perhaps with bowed strings

2 Other Polarizations

Obviously, the string does not only vibrate in a single transversal polarization, but also in the

- other transversal polarization
  - termination impedance is different in different polarizations ⇒ different decay times ⇒ two-stage decay!
- longitudinal polarization
  - basically the same as transversal vibration, but the propagation velocity is different (see R&F: Sec. 2.14). Typically $c_L \gg c$
  - might connect to the instrument body (and become audible)
- rotational polarization
  - usually negligible, except perhaps with bowed strings

The total string vibration is a superposition of all vibrations in different polarizations.

2 Effect of Nonlinearities

Some nonlinear effects in vibrating strings:

- tension modulation with large amplitudes
  - string tension varies during vibration

Some nonlinear effects in vibrating strings:

- tension modulation with large amplitudes
  - string tension varies during vibration
  - causes initial pitch glide

2 Effect of Nonlinearities

Some nonlinear effects in vibrating strings:

- tension modulation with large amplitudes
  - string tension varies during vibration
  - causes initial pitch glide
- stick-slip coupling between bow and string
  - `http://www.youtube.com/watch?v=KPpBvHXYWz4`

2 Effect of Nonlinearities

Some nonlinear effects in vibrating strings:

- tension modulation with large amplitudes
  - string tension varies during vibration
  - causes initial pitch glide
- stick-slip coupling between bow and string
  - `http://www.youtube.com/watch?v=KPpBvHXYWz4`
- hammer nonlinearity (in the piano)
  - hammer seems harder when played with greater intensity
  - timbre changes as a function of key velocity

2 Effect of Nonlinearities

Some nonlinear effects in vibrating strings:

- tension modulation with large amplitudes
  - string tension varies during vibration
  - causes initial pitch glide
- stick-slip coupling between bow and string
  - `http://www.youtube.com/watch?v=KPpBvHXYWz4`
- hammer nonlinearity (in the piano)
  - hammer seems harder when played with greater intensity
  - timbre changes as a function of key velocity
- other nonlinear effects in musical instruments discussed in Nonlinear physics of musical instruments by Fletcher.

3

Rectangular Membranes

3 General

Basically, membranes can be seen as 2D-extensions of ideal strings.

3 General

Basically, membranes can be seen as 2D-extensions of ideal strings.
In the following, we will study the free vibration of rectangular and circular membranes in the lossless case (R&F:chap 3).
In reality, losses and nonlinear effects also have a strong impact on the vibration.

3 Rectangular Membrane

Let's consider a perfectly homogeneous and flexible sheet with:

- mass per unit area σ [kg/m^2]
- tension T [N/m] applied via the edges
- positioned at the *xy*-plane (*z*-axis denotes displacement)

3 Rectangular Membrane II

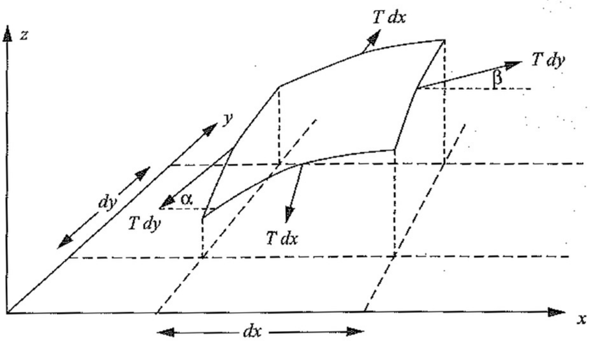


Fig. 3.1. Forces on a rectangular membrane element.

3 Wave Equation for the Rectangular Membrane

Similarly to the string, if $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \ll 1 \Rightarrow$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right), \text{ where } c^2 = \frac{T}{\sigma}$$

3 Wave Equation for the Rectangular Membrane

Similarly to the string, if $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \ll 1 \Rightarrow$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right), \text{ where } c^2 = \frac{T}{\sigma}$$

The parenthesis term is the Laplace-operator, denoted $\bigtriangledown^2$

3 Wave Equation for the Rectangular Membrane

Similarly to the string, if $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \ll 1 \Rightarrow$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right), \text{ where } c^2 = \frac{T}{\sigma}$$

The parenthesis term is the Laplace-operator, denoted $\bigtriangledown^2$, so the wave equation can be given in a more compact form:

$$\ddot{z} = c^2 \bigtriangledown^2 z \tag{8}$$

3 Vibration Equation

The solution to Eq. (8) can be given as (R&F: pp.66-67):

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi}{L_x}x\right) \sin\left(\frac{n\pi}{L_y}y\right) \left(M \sin(\omega_{mn}t) + N \cos(\omega_{mn}t)\right) \tag{9}$$

which consists of

The solution to Eq. (8) can be given as (R&F: pp.66-67):

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi}{L_x}x\right) \sin\left(\frac{n\pi}{L_y}y\right) \left(M \sin(\omega_{mn}t) + N \cos(\omega_{mn}t)\right) \tag{9}$$

which consists of

- **sum over *two* types of modes**

3 Vibration Equation

The solution to Eq. (8) can be given as (R&F: pp.66-67):

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi}{L_x}x\right) \sin\left(\frac{n\pi}{L_y}y\right) \left(M \sin(\omega_{mn}t) + N \cos(\omega_{mn}t)\right) \tag{9}$$

which consists of

- sum over *two* types of modes
- **spatial modes in x-direction**

3 Vibration Equation

The solution to Eq. (8) can be given as (R&F: pp.66-67):

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi}{L_x} x\right) \sin\left(\frac{n\pi}{L_y} y\right) \left(M \sin(\omega_{mn} t) + N \cos(\omega_{mn} t)\right) \tag{9}$$

which consists of

- sum over *two* types of modes
- spatial modes in x-direction
- **spatial modes in y-direction**

The solution to Eq. (8) can be given as (R&F: pp.66-67):

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi}{L_x}x\right) \sin\left(\frac{n\pi}{L_y}y\right) \left(M \sin(\omega_{mn}t) + N \cos(\omega_{mn}t)\right) \tag{9}$$

which consists of

- sum over *two* types of modes
- spatial modes in x-direction
- spatial modes in y-direction
- **temporal vibration**

3 Vibration Equation

The solution to Eq. (8) can be given as (R&F: pp.66-67):

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi}{L_x} x\right) \sin\left(\frac{n\pi}{L_y} y\right) \left(M \sin(\omega_{mn} t) + N \cos(\omega_{mn} t)\right) \tag{9}$$

which consists of

- sum over *two* types of modes
- spatial modes in x-direction
- spatial modes in y-direction
- temporal vibration

Both spatial directions have their own modes!

3 Eigenfrequencies and -modes

The eigenfrequencies are given as

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

where L_x and L_y are the dimensions of the membrane and $m, n = 1, 2, 3, ...$

3 Eigenfrequencies and -modes

The eigenfrequencies are given as

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

where L_x and L_y are the dimensions of the membrane and $m, n = 1, 2, 3, \ldots$ The eigenmodes depend on both n and m:

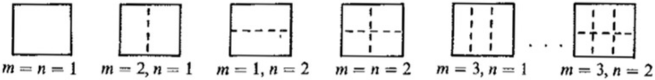


Fig. 3.2. Some normal modes of a rectangular membrane.

3 Eigenfrequencies and -modes

The eigenfrequencies are given as

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

where L_x and L_y are the dimensions of the membrane and $m, n = 1, 2, 3, ...$ The eigenmodes depend on both n and m:

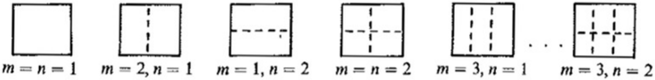


Fig. 3.2. Some normal modes of a rectangular membrane.

Animation:

`https://www.acs.psu.edu/drussell/Demos/rect-membrane/rect-mem.html`

3 Eigenfrequencies and -modes II

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

Let's consider a rectangular membrane with $T = 100$, $\sigma = 0.01$, and $L_x = L_y = 1$. What are the eigenfrequencies f_{11}, f_{12}, and f_{21}?

3 Eigenfrequencies and -modes II

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

Let's consider a rectangular membrane with $T = 100$, $\sigma = 0.01$, and $L_x = L_y = 1$. What are the eigenfrequencies f_{11}, f_{12}, and f_{21}?

- $f_{11} = 50\sqrt{(2)} \approx 71$ Hz

3 Eigenfrequencies and -modes II

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

Let's consider a rectangular membrane with $T = 100$, $\sigma = 0.01$, and $L_x = L_y = 1$. What are the eigenfrequencies f_{11}, f_{12}, and f_{21}?

- $f_{11} = 50\sqrt{(2)} \approx 71$ Hz
- $f_{12} = f_{21} = 50\sqrt{(5)} \approx 112$ Hz

3 Eigenfrequencies and -modes II

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

Let's consider a rectangular membrane with $T = 100$, $\sigma = 0.01$, and $L_x = L_y = 1$. What are the eigenfrequencies f_{11}, f_{12}, and f_{21}?

- $f_{11} = 50\sqrt{(2)} \approx 71$ Hz
- $f_{12} = f_{21} = 50\sqrt{(5)} \approx 112$ Hz
- $\Rightarrow$ not in a harmonic relation, inharmonic sound!

3 Eigenfrequencies and -modes II

$$f_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2} \qquad \text{(R\&F:(3.4))}$$

Let's consider a rectangular membrane with $T = 100$, $\sigma = 0.01$, and $L_x = L_y = 1$. What are the eigenfrequencies f_{11}, f_{12}, and f_{21}?

- $f_{11} = 50\sqrt{(2)} \approx 71$ Hz
- $f_{12} = f_{21} = 50\sqrt{(5)} \approx 112$ Hz
- $\Rightarrow$ not in a harmonic relation, inharmonic sound!
- generally no distinct pitch sensation, although one can be obtained by carefully selecting the dimensions and damping some of the modes

3 Eigenfrequencies and -modes III

Similarly to a string, the eigenfrequencies are proportional to the square root of the tension, but they are more densely located in the membrane

3 Eigenfrequencies and -modes III

Similarly to a string, the eigenfrequencies are proportional to the square root of the tension, but they are more densely located in the membrane

- in a square membrane, some of the modes have the same eigenfrequency
  - these are called *degenerate modes*

See again:

`https://www.acs.psu.edu/drussell/demos/membranesquare/square.html`

4

Circular Membranes

4 Circular Membranes

Consider a circular membrane with radius R

- displacement given as $z(r, \phi)$ (in polar coordinates)

4 Circular Membranes

Consider a circular membrane with radius R

- displacement given as $z(r, \phi)$ (in polar coordinates)
- fixed at the boundary $z(R, \phi) = 0$

4 Circular Membranes

Consider a circular membrane with radius R

- displacement given as $z(r, \phi)$ (in polar coordinates)
- fixed at the boundary $z(R, \phi) = 0$

From the wave equation of the circular membrane (R&F:(3.7)) one can evaluate the eigenfrequencies:

$$f_{mn} = \frac{c}{2\pi R} J_{mn}, \tag{10}$$

where J_{mn} is the nth zero of the mth Bessel function

4 Circular Membranes

Consider a circular membrane with radius R

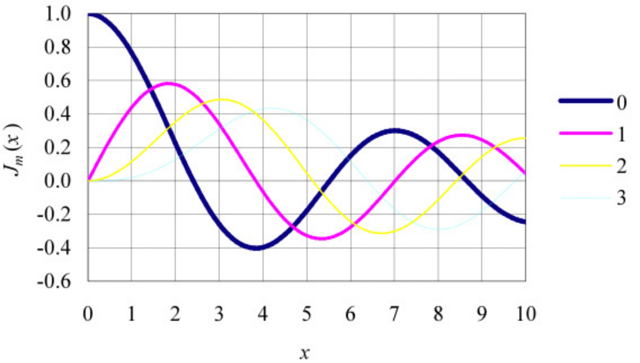
- displacement given as $z(r, \phi)$ (in polar coordinates)
- fixed at the boundary $z(R, \phi) = 0$

From the wave equation of the circular membrane (R&F:(3.7)) one can evaluate the eigenfrequencies:

$$f_{mn} = \frac{c}{2\pi R} J_{mn}, \tag{10}$$

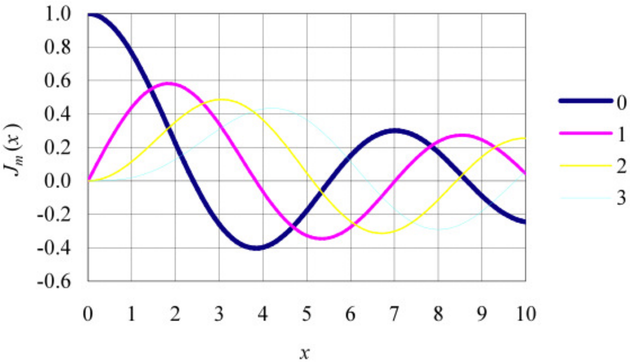
where J_{mn} is the nth zero of the mth Bessel function (in practice, check e.g. from **this table**).

4 Some of the lowest-order Bessel Functions



4 Some of the lowest-order Bessel Functions

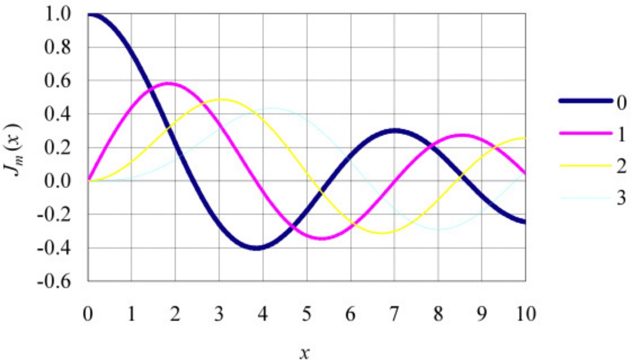
Circular Membranes
Bessel's function
Mode Patterns



$$f_{mn} = \frac{c}{2\pi R} J_{mn},$$

What is the f_{21} for a circular membrane with $c = 100\ \frac{\text{m}}{\text{s}}$, $R = \frac{1}{2\pi}$ m?

4 Some of the lowest-order Bessel Functions



$$f_{mn} = \frac{c}{2\pi R} J_{mn},$$

What is the f_{21} for a circular membrane with $c = 100\ \frac{\text{m}}{\text{s}}$, $R = \frac{1}{2\pi}$ m?
- Approximately 514 Hz.

4 Mode Patterns

Some of the lowest modal patterns given in (R&F:fig.3.6):

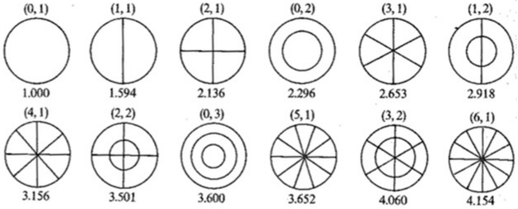


Fig. 3.6. First 12 modes of an ideal membrane. The mode designation (m, n) is given above each figure and the relative frequency below. To convert these to actual frequencies, multiply by $(2.405/2\pi a)\sqrt{T/\sigma}$, where a is the membrane radius.

(the number below each mode expresses the frequency ratio to the fundamental)

4 Mode Patterns

Some of the lowest modal patterns given in (R&F:fig.3.6):

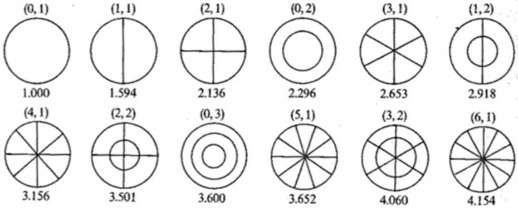


Fig. 3.6. First 12 modes of an ideal membrane. The mode designation (m, n) is given above each figure and the relative frequency below. To convert these to actual frequencies, multiply by $(2.405/2\pi a)\sqrt{T/\sigma}$, where a is the membrane radius.

(the number below each mode expresses the frequency ratio to the fundamental) $\Rightarrow$ again, an inharmonic spectrum!

4 Mode Patterns

Some of the lowest modal patterns given in (R&F:fig.3.6):

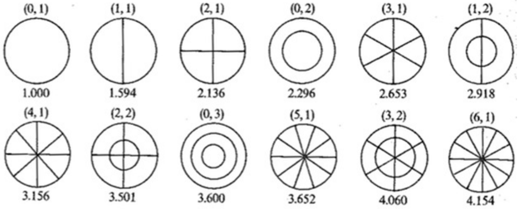


Fig. 3.6. First 12 modes of an ideal membrane. The mode designation (m, n) is given above each figure and the relative frequency below. To convert these to actual frequencies, multiply by $(2.405/2\pi a)\sqrt{T/\sigma}$, where a is the membrane radius.

(the number below each mode expresses the frequency ratio to the fundamental) $\Rightarrow$ again, an inharmonic spectrum! See animation at `https://www.acs.psu.edu/drussell/Demos/MembraneCircle/Circle.html` or `http://www.falstad.com/circosc/`