# Exercise session 3

① a) Prove the reverse triangle inequality

$$\big||z|-|w|\big| \leq |z-w|$$

for any $z, w \in \mathbb{C}$.

Hint: $z = (z-w)+w$

b) Show that the set

$$\overline{\Delta(z_0, r)} = \{z \in \mathbb{C} ; |z-z_0| \leq r\}$$

is closed.

Hint: 1a is useful to show that the complement is open.

Solution: a) $|z| = |(z-w)+w| \leq |z-w| + |w|$ by the ordinary triangle inequality. We get

$$|z|-|w| \leq |z-w|.$$

Let $z$ and $w$ switch roles to get

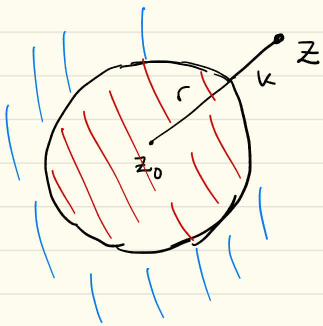
$$|w|-|z| \leq |z-w|$$

$$\Longrightarrow \big||z|-|w|\big| \leq |z-w| \quad \otimes$$

b)


We want to show that the set (red in figure) $\overline{\Delta(z_0,r)} = \{z \in \mathbb{C};\ |z-z_0| \le r\}$ is closed. We do this by showing that the complement (blue in figure) $\{z \in \mathbb{C};\ |z-z_0| > r\}$ is open

Pick $z$ such that $|z-z_0| = r+k$ $\quad (k>0)$




<u>Claim</u>:

$\{w \in \mathbb{C};\ |w-z| < \frac{k}{2}\}$ is contained in $\{z \in \mathbb{C};\ |z-z_0| > r\}$

This is indeed the case since

$$|w-z_0| = |(w-z)+(z-z_0)| \ge \big||w-z|-|z-z_0|\big| =$$

$$= r+k-|w-z| \ge r+k-\frac{k}{2} > r$$

$r+k > \frac{k}{2}$

Hence $\{z \in \mathbb{C};\ |z-z_0| > r\}$ is open and therefore

$$\overline{\Delta(z_0,r)} = \{z \in \mathbb{C};\ |z-z_0| \le r\}$$

is closed.

(2) Compute $\lim_{n\to\infty} z_n$ when

a) $z_n = i^{n!} + 2^{-n}$

b) $z_n = 2^{-n+i\sqrt{n}}$

c) $z_n = \sqrt[n]{z}$, for $z \in \mathbb{C}$.

Solution: a) Notice that $i^4 = 1$ and that

$n!$ is an integer multiple of 4 when $n \geq 4$.

Therefore $z_n = 1 + 2^{-n}$ when $n \geq 4$.

We now calculate

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} 1 + 2^{-n} = 1$$

b) $$z_n = 2^{-n+i\sqrt{n}} = 2^{-n} \cdot 2^{i\sqrt{n}} = 2^{-n} \cdot \underbrace{e^{(\ln 2)\sqrt{n}\, i}}_{= \cos(\sqrt{n}\ln 2) + i\sin(\sqrt{n}\ln 2)}$$

Therefore $|z_n| = 2^{-n}$ and since

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} 2^{-n} = 0$$

also $\lim_{n\to\infty} z_n = 0$.

c) $$z_n = \sqrt[n]{z} = e^{\frac{1}{n}(\mathrm{Log}(z))} =$$

$$= e^{\frac{1}{n}\ln|z| + i\frac{\mathrm{Arg}(z)}{n}} =$$

$$= e^{\frac{1}{n}\ln|z|}\left(\cos\left(\frac{\mathrm{Arg}(z)}{n}\right) + i\sin\left(\frac{\mathrm{Arg}(z)}{n}\right)\right)$$

$\lim\limits_{n\to\infty} \frac{\ln|z|}{n} = 0$ and also $\lim\limits_{n\to\infty} \frac{\mathrm{Arg}(z)}{n} = 0$

Since exp, cos and sin are continuous we get

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \sqrt[n]{z} = e^0 \cdot (1 + i0) =$$
$$= 1$$

(unless $z = 0$. If $z = 0$ then $\sqrt[n]{z}$ is undefined!)

3 A complex sequence is defined recursively by $z_1 = 0$, $z_2 = i$, and $z_n = \frac{z_{n-1} + z_{n-2}}{2}$ for $n \geq 3$. Show that

$$z_n = \frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{n-1}\right)$$

when $n \geq 2$.

Calculate $\lim\limits_{n\to\infty} z_n$.

Solution: We verify the formula

$$z_n = \frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{n-1}\right) \quad \text{for } n \geq 2$$

by induction.

Base step: $n=2$

$$\frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{2-1}\right) = \frac{2i}{3}\left(\frac{3}{2}\right) = i$$

OK!

$n=3$

$$\frac{z_1 + z_2}{2} = \frac{i}{2}$$

$$\frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{3-1}\right) = \frac{2i}{3}\left(1 - \frac{1}{4}\right) = \frac{2i}{3} \cdot \frac{3}{4} = \frac{i}{2}$$

Induction step

Assume the formula

$$z_k = \frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{k-1}\right) \text{ is true}$$

for $2 \leq k \leq p$.

$$z_{p+1} = \frac{z_p + z_{p-1}}{2} = \frac{2i}{2 \cdot 3}\left(\left(1 - \left(-\frac{1}{2}\right)^{p-1}\right) + \left(1 - \left(-\frac{1}{2}\right)^{p-2}\right)\right)$$

$$= \frac{2i}{3} \cdot \frac{2 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{p-2} - \left(-\frac{1}{2}\right)^{p-2}}{2} =$$

$$= \frac{2i}{3}\left(\frac{2 - \frac{1}{2}\left(-\frac{1}{2}\right)^{p-2}}{2}\right) = \frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{2}\left(-\frac{1}{2}\right)^{p-2}\right)$$

$$= \frac{2i}{3}\left(1 - \left(-\frac{1}{2}\right)^{p}\right)$$

OK!

So the formula is true for $n \geq 2$ by induction.

We get

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{2i}{3}\left(1-\left(-\frac{1}{2}\right)^{n-1}\right) = \frac{2i}{3}.$$

$\otimes$

(4) Compute:

a) $\lim_{z\to i} \frac{z^4+1}{z+i}$

b) $\lim_{z\to -i} \frac{z^4-1}{z+i}$

c) $\lim_{z\to 2i} \frac{z^2-iz+2}{z^2+4}$

<u>Warning</u>: You cannot use l'Hospitals rule (yet)!

<u>Solution</u>: a) Since polynomials are continuous we can just plug in $z=i$ (unless the numerator is zero)

$$\lim_{z\to i} \frac{z^4+1}{z+i} = \frac{i^4+1}{i+i} = \frac{2}{2i} = \frac{1}{i} = -i$$

b) $\displaystyle\lim_{z\to -i} \frac{z^4-1}{z+i} = ?$

Notice that the numerator $-i+i=0$. However also $i^4-1 = 1-1 = 0$.

$$\frac{z^4-1}{z+i} = \frac{(z^2+1)(z^2-1)}{z+i} = \frac{(z-i)\cancel{(z+i)}(z^2-1)}{\cancel{z+i}}.$$

Therefore $\displaystyle\lim_{z\to -i} \frac{z^4-1}{z+i} = \lim_{z\to -i} (z-i)(z^2-1) =$

$= -2i\,(i^2-1) = 4i.$

c) $\displaystyle\lim_{z\to 2i} \frac{z^2-iz+2}{z^2+4}$

Notice $(2i)^2+4 = -4+4=0$ and

$(2i)^2 - i(2i)+2 = -4+2+2=0$

So $z^2-iz+2 = (z-2i)(z+i)$

and $z^2+4 = (z-2i)(z+2i)$

Therefore $\displaystyle\lim_{z\to 2i} \frac{z^2-iz+2}{z^2+4} = \lim_{z\to 2i} \frac{(z+i)(z-2i)}{(z+2i)(z-2i)} =$

$\displaystyle = \lim_{z\to 2i} \frac{z+i}{z+2i} = \frac{3i}{4i} = \frac{3}{4}.$