Exercise sheet 4

1. For each of the following functions determine the largest open set in which it is analytic and calculate its derivative

a) $f(z) = z^3(1+z)^6$

b) $g(z) = \frac{z+1}{z-2i}$

c) $h(z) = \left(\frac{z-1}{z^3-27}\right)^4$

<u>Solution</u>: a) $f(z)$ is a polynomial and hence analytic in the whole plane $\mathbb{C}$ (that is f is entire)

$$f'(z) = 3z^2(1+z)^6 + 6z^3(1+z)^5 =$$
$$= 3z^2(1+z)^5((1+z)+2z) = 3z^2(1+z)^5(1+3z)$$

b) $g(z)$ is analytic if $z-2i \neq 0$. That is when $z \neq 2i$

$$g'(z) = \frac{(z-2i)-(z+1)}{(z-2i)^2} = \frac{-1-2i}{(z-2i)^2}$$

c) $h(z)$ is analytic if $z^3-27 \neq 0$

$$z^3-27 = (z-3)(z^2+3z+9) = 0$$
$$z^2+3z+9 = \left(z+\frac{3}{2}\right)^2 + 9 - \frac{9}{4} = 0$$
$$\left(z+\frac{3}{2}\right)^2 = -\frac{27}{4}$$

$\Longrightarrow z^3-27=0$ iff $z_1 = 3$, $z_2 = -\frac{3}{2} + i\frac{3\sqrt{3}}{2}$, or $z_3 = -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$

So $h(z)$ is analytic in $\mathbb{C} \setminus \{3, -\frac{3}{2}+i\frac{3\sqrt{3}}{2}, -\frac{3}{2}-i\frac{3\sqrt{3}}{2}\}$

and $h'(z) = 4\left(\frac{z-1}{z^3-27}\right)^3 \cdot \frac{d}{dz}\left(\frac{z-1}{z^3-27}\right) =$

$$= 4\left(\frac{z-1}{z^3-27}\right)^3 \cdot \frac{z^3-27-3z^2(z-1)}{(z^3-27)^2} =$$

$$= \frac{4(z-1)^3(-2z^3+3z^2-27)}{(z^3-27)^5}$$

2. Let $f(z) = 1-y^2+i(2xy-y^2)$. Locate all points z at which f is complex differentiable, and determine $f'(z)$ for each such point.

Solution: We will use the Cauchy-Riemann equations. First $f(z) = u(x,y)+iv(x,y)$ where

$$u(x,y) = 1-y^2 \qquad \text{and} \qquad v(x,y) = 2xy-y^2$$

So $\quad u_x = 0 \quad , \quad v_y = 2x-2y,$

$$v_x = 2y \quad , \text{ and } \quad u_y = -2y$$

These are all continuous so if CR-equations holds at a point then f is complex differentiable there.

So $\quad u_x = v_y \quad$ gives $\quad 0 = 2x - 2y$ and

$\quad v_x = -u_y \quad$ gives $\quad 2y = 2y$

The second equation is always satisfied.
The first holds if $\quad y = x$.

$\Longrightarrow$ The Cauchy-Riemann equations hold on the line $y = x$.

$f'(z) = u_x(z) + i\, v_x(z) = 0 + i\, 2y \quad$ for these points.

(Sanity check?: $f'(z) = -i\left(u_y(z) + i\, v_y(z) \right)$

$$= v_y(z) - i\, u_y(z) = (2x - 2y) - i(-2y) = \underset{x=y}{=} 0 + i\, 2y \quad)$$

(3) Let $U \subseteq \mathbb{C}$ be an open set. Assume that $h : U \to \mathbb{R}$ has continuous partial derivatives of 1st and 2nd order and satisfy Laplace's equation $h_{xx} + h_{yy} = 0$ in U. Show that

$$f(z) = h_x(x,y) - i h_y(x,y) \text{ is analytic}$$

in U. (A function solving Laplace's equation is called harmonic.)

<u>Solution</u>: We will use the Cauchy-Riemann equations. Putting $u = h_x$ and $v = -h_y$ we see that

$u_x = h_{xx}$, $u_y = h_{xy}$, $v_x = -h_{yx}$, and $v_y = -h_{yy}$ are continuous. Also the Cauchy-Riemann equations are easily verified since

$u_x = v_y$ is equivalent to $h_{xx} = -h_{yy}$

$\iff h_{xx} + h_{yy} = 0$ ✓

and since $h_{xy} = h_{yx}$ we get

$u_y = -v_x$. Therefore $f(z) = h_x(x,y) - i h_y(x,y)$ is analytic

(We started with a harmonic function and produced an analytic function)

④ Assume that $U \subseteq \mathbb{C}$ is an open set and that $f: U \to \mathbb{C}$ is analytic. Let $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ and assume that u and v has continuous partial derivatives of 1st and 2nd order. (We will later see that the assumptions on u and v are automatically true.) Show that u and v are harmonic (that is solve Laplace's equation in Exercise 3.)

<u>Solution</u>: Since f is analytic u and v satisfies the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. Now

$$u_{xx} = v_{yx} \quad \text{and} \quad u_{yy} = -v_{xy}$$

so $$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

since $v_{xy} = v_{yx}$.

Also $v_{yy} = u_{xy}$ and $v_{xx} = -u_{yx}$ which in the same way gives

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

(We started with an analytic function and produced two harmonic functions.)