Microeconomic Theory II
Helsinki GSE 2023
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Problem Set 1, Due November 15, 2023

1. Three students are looking for an apartment to rent and share. Unfortunately, the students have different preferences. Student 1 ranks the apartments in the decreasing order of the distance to Economicum. Student 2 ranks the apartments in the decreasing order of monthly rent. Student 3 ranks them in the increasing order of floor area in square meters. The students have identified 5 potential apartments listed arbitrarily as a, b, c, d, e and no student is indifferent between any pair of these apartments. To come up with a choice, they conduct a majority vote first between a and b. The winner of the first vote enters the second voting stage against c. The winner in the second stage meets d in a third stage vote, and finally the winner of the third stage meets e in the final stage. The winning apartment is chosen. Assume also that the students vote sincerely at any stage (i.e. they vote for the alternative that they like better).

   (a) Is the preference order induced by pairwise majority votes between alternatives complete and transitive?

   (b) Does the final choice depend on the order of votes, i.e. is the outcome always the same for any permutation of the apartments in the voting protocol? (For example, we could start with b against d, then winner meets a, then winner meets e and finally winner meets e.)

   (c) Could any of the students ever gain by voting strategically (i.e. voting for the worse alternative in some stage)?

2. Three other students come up with a different method for choosing the apartment to share. They take turns eliminating alternatives so that Student 1 eliminates her worst alternative, then Student 2 eliminates her worst alternative, then Student 3, then Student 1 etc. until a single alternative remains.

   (a) Does this method always result in a Pareto-efficient last remaining alternative?

(b) Would it ever be advantageous for any of the students to eliminate an alternative that is not the worst alternative in some stage?

3. Three graduate students $\{1, 2, 3\}$ rely on scholarships to fund their studies. Scholarship income is unfortunately stochastic and the students prefer smooth consumption. Suppose that the students have the same quadratic utility functions and their scholarship income y_i has mean μ_i and variance σ_i for student $i \in \{1, 2, 3\}$ (and assume that the support of all scholarship income distributions is on the increasing part of the utility function so that it is optimal to consume the entire scholarship income).

   (a) Denote the consumption of student $\imath$ by c_i, and let
   $$u(c_i) = ac_i - bc_i^2.$$
   What is the expected utility of student i if $y_i = c_i$, i.e. each student just consumes her own scholarship income?

   (b) Suppose that the students' incomes are statistically independent. If the students pool their incomes and share the pooled income equally for consumption, then
   $$c_i = \frac{1}{n} \sum_i y_i \text{ for all } i.$$
   Find a condition in terms of the μ_i and σ_i ensuring that all students have an incentive to participate in the pooling (rather than staying on their own as in part a).

   (c) Total surplus amongst the students in the pool is the sum of utilities over all students. The marginal contribution of a student to the pool is the increase in the total surplus that results from her addition to the pool. Compute the marginal contribution of each student to a pool of n identical students (i.e. $\mu_i = \mu, \quad \sigma_i = \sigma$ for all i).

4. A patient sister s negotiates with her impatient brother b over how to share their monthly candy portion so that the monthly shares of the two children satisfy $x_{s,t} + x_{b,t} = 1$ over the coming 12 months, $t \in \{1, 2, ..., 12\}$. Let $1 \geq \beta_s > \beta_b > 0$ be the children's (monthly) discount factors. Suppose that the intertemporal utility function of

child $i \in \{s, b\}$ is given by

$$U^i(x_{i,1}, ..., x_{i,12}) = \sum_{t=1}^{12} \beta_i^t x_{i,t}.$$

(a) What are the Pareto-efficient allocations?

(b) Suppose that the children have the option to ask their parents to allocate $\frac{1}{2}$ to each child in each month. What are the Pareto efficient allocations that are acceptable to both children?

(c) How does your answer change if $U^i(x_{i,1}, ..., x_{i,12}) = \sum_{t=1}^{12} \beta_i^t \ln(x_{i,t})$?

5. Consider housing allocations in a society.

   (a) Show that adding an agent and the house that she occupies may make some of the original agents worse off in the equilibrium of the new society relative to the equilibrium of the original society.

   (b) Suppose 5 unoccupied houses are located on a line and each of the 5 agents cares about the house and her nearest neighbor (or neighbors if not at the end of the line). Describe a process for finding a Pareto-efficient allocation of the houses to the agents.

6. Consider an economy with n agents. Let X be the set of alternatives available in this economy. For each pair $(x, y) \in X \times X$, define the variable d_i for each $i \in \{1, ..., n\}$ as follows:

$$d_i = \begin{cases} 1 \text{ if } x \succ_i y, \\ 0 \text{ if } x \sim_i y, \\ -1 \text{ if } y \succ_i x. \end{cases}$$

A social choice function is a function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ (with the same interpretation as above). Let $d = (d_1, ..., d_n)$. The majority decision rule is defined as follows:

$$f(d_1, ..., d_n) = \begin{cases} 1 \text{ if } \Sigma_{i=1}^n d_i > 0, \\ 0 \text{ if } \Sigma_{i=1}^n d_i = 0, \\ -1 \text{ if } \Sigma_{i=1}^n d_i < 0. \end{cases}$$

Let $n^+(d) = \#\{i \text{ such that } d_i = 1\}$ and $n_-(d) = \#\{i \text{ such that } d_i = -1\}$. A social choice function is said to be *anonymous* if $f(d) = f(d')$ whenever $n^+(d) = n^+(d')$ and $n_-(d) = n_-(d')$. In other words, the rule treats all individuals in the same manner. A social choice function is *neutral* if $f(-d) = -f(d)$. A social choice function is *responsive* if $f(d) \geq 0$ and $d' > d$ imply that $f(d') = 1$.

(a) Show that the majority rule is anonymous, neutral and responsive.

(b) Show that whenever f is anonymous and neutral, $n^{+}(d) = n_{-}(d)$ implies that $f(d) = 0$.

(c) Prove that whenever f is anonymous, neutral and responsive, it is given by the majority rule.

7. Consider the single-dimensional spatial model where the set of alternatives is given by the interval $X = [0, 1]$ and there are an odd number of voters $i \in \{1, ..., n\}$. Each voter has rational preferences over X. Assume further that for each i, there is an ideal alternative $x_i^* \in [0, 1]$ and that the preferences are single-peaked, i.e.

$$x < x' < x_i^* \implies x_i^* \succ x' \succ x \text{ and } x > x' > x_i^* \implies x_i^* \succ x' \succ x.$$

(a) Show that the societal preference induced by majority voting between pairs of alternatives is complete and transitive.

(b) Show that the ideal point of the median voter (i.e. the median of the set $\{x_1^*, ..., x_n^*\}$) is strictly preferred to any other alternative in the social preference induced by majority voting.

(c) Is the median voter a dictator in the sense of Arrow's theorem?