Solutions: Problem Set 1

1. Three students are looking for an apartment to rent and share. Unfortunately, the students have different preferences. Student 1 ranks the apartments in the decreasing order of the distance to Economicum. Student 2 ranks the apartments in the decreasing order of monthly rent. Student 3 ranks them in the increasing order of floor area in square meters. The students have identified 5 potential apartments listed arbitrarily as a, b, c, d, e and no student is indifferent between any pair of these apartments. To come up with a choice, they conduct a majority vote first between a and b. The winner of the first vote enters the second voting stage against c. The winner in the second stage meets d in a third stage vote, and finally the winner of the third stage meets e in the final stage. The winning apartment is chosen. Assume also that the students vote sincerely at any stage (i.e. they vote for the alternative that they like better).

a) Is the preference order induced by pairwise majority votes between alternatives complete and transitive?

Answer: The preference order is complete but not necessarily transitive. This situation is called 'The Condorcet Paradox'[1] and in this situation collective preferences can be cyclic, even if the preferences of individual agents are not cyclic. Thus, transitivity doesn't hold in general. However, $\succeq^M$ is a complete social preference relation, since for any two allocations $a_j, a_{j'}$ we have $\#\{i|a_j \succeq_i a_{j'}\} \geq \#\{i|a_{j'} \succeq_i a_j\}$ or $\#\{i|a_j \succeq_i a_{j'}\} \leq \#\{i|a_{j'} \succeq_i a_j\}$.

Example:

1	2	3
a	c	b
b	a	c
c	b	a
d	e	d
e	d	e

We see that $a \succ^M b$, $b \succ^M c$ and $c \succ^M a$, which violates the transitivity requirement on social preferences.

b) Does the final choice depend on the order of votes, i.e. is the outcome always the same for any permutation of the apartments in the voting protocol? (For example, we could start with b against d, then winner meets a, then winner meets e and finally winner meets e.)

Answer: The final choice depends on the order of votes because transitivity is violated. Consider the example above. If we compare options in alphabetical order we end up choosing

[1] See page 796 in MWC

c. However, if we compare options such that we start by comparing b to c and then proceed in alphabetical order we end up choosing a.

c) Could any of the students ever gain by voting strategically (i.e. voting for the worse alternative in some stage)?

Answer: Yes they could. Consider the example above when the students vote in alphabetical order. The student number 1 would benefit from voting b instead of a in the first round so that b would survive the first elimination and then beat c in the next round. Then the students would end up choosing b instead of c which would be a better option for student number 1.

2. Three other students come up with a different method for choosing the apartment to share. They take turns eliminating alternatives so that Student 1 eliminates her worst alternative, then Student 2 eliminates her worst alternative, then Student 3, then Student 1 etc. until a single alternative remains.

a) Does this method always result in a Pareto-efficient last remaining alternative?

Answer: Let's choose an arbitrary x^* and assume that it Pareto dominates the last remaining alternative x. Then it must be that $x^* \succeq_i x$ for all i and $x^* \succ_i x$ at least for one i. However, if x^* was not the last remaining alternative, it means that one of the students eliminated it. This means that (at least) for one of the students x^* was the worst alternative when x still remained in set of alternatives, implying $x \succ_i x^*$. Thus, x^* cannot Pareto dominate and thus the method always results in a Pareto-efficient last remaining alternative.

b) Would it ever be advantageous for any of the students to eliminate an alternative that is not the worst alternative in some stage?

Answer: Yes. Consider the example below. If Student 1 and Student 3 eliminate always their worst alternative, then Student 2 has an incentive to eliminate b rather than d when their turn comes. By doing that, the students end up choosing c instead of b which is better for Student 2.

Example:

1	2	3
a	c	b
b	a	c
c	b	e
d	e	d
e	d	a

3. Three graduate students $\{1,2,3\}$ rely on scholarships to fund their studies. Scholarship income is unfortunately stochastic and the students prefer smooth consumption. Suppose that the students have the same quadratic utility functions and their scholarship income y_i has mean μ_i and variance σ_i for student $i \in \{1,2,3\}$ (and assume that the support of all scholarship income distributions is on the increasing part of the utility function so that it is optimal to consume the entire scholarship income).

a) Denote the consumption of student i by c_i, and let

$$u(c_i) = ac_i - bc_i^2$$

What is the expected utility of student i if $y_i = c_i$, i.e. each student just consumes her own scholarship income?

Answer: The expected utility of a quadratic function is :

$$E[u(c_i)] = aE[c_i] - bE[c_i^2]$$

$$E[u(c_i)] = aE[c_i] - b(VAR(c_i) + (E[c_i])^2)$$

$$E[u(c_i)] = aE[c_i] - b(E[c_i])^2 - bVAR[c_i]$$

Because

$$VAR[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = VAR[X] + (E[X])^2$$

As $c_i = y_i$ we get:

$$E[u(y_i)] = aE[y_i] - b(E[y_i])^2 - bVAR[y_i]$$

And we know that $E[y_i] = \mu_i$ and $VAR[y_i] = \sigma_i$. Then the expected utility is:

$$E[u(\mu_i, \sigma_i)] = a\mu_i - b\mu_i^2 - b\sigma_i$$

b) Suppose that the students' incomes are statistically independent. If the students pool their incomes and share the pooled income equally for consumption, then

$$c_i = \frac{1}{n}\sum_i y_i \text{ for all i}$$

Find a condition in terms of the μ_i and σ_i ensuring that all students have an incentive to participate in the pooling (rather than staying on their own as in part a).

Answer: For all students to have an incentive to participate in the pooling, their expected utility from the pooling has to be greater (or equal) to the expected utility they get individually.

Therefore, lets first calculate their expected utility from the pooling. We know from part a) that the expected utility of a quadratic function is as follows:

$$E[u^P(c_i)] = aE[c_i] - b(E[c_i])^2 - bVAR[c_i]$$

As $c_i = \frac{1}{n}\sum_i y_i$ we need to know $E[\frac{1}{n}\sum_i y_i]$ and $VAR[\frac{1}{n}\sum_i y_i]$. Because the students' incomes are statistically independent, we can take the sum out of the expected value

$$E[\frac{1}{n}\sum_i y_i] = \frac{1}{n}\sum_i E[y_i] = \frac{1}{n}\sum_i \mu_i$$

As the students' incomes are statistically independent, the variance of $\frac{1}{n}\sum_i y_i$ is:

$$VAR[\frac{1}{n}\sum_i y_i] = (\frac{1}{n})^2 \sum_i VAR[y_i] = \frac{1}{n^2}\sum_i \sigma_i$$

Thus, the expected value is:

$$E[u^P(\mu_i, \sigma_i)] = \frac{a}{n}\sum_i \mu_i - \frac{b}{n^2}(\sum_i \mu_i)^2 - \frac{b}{n^2}\sum_i \sigma_i$$

For the students to have an incentive to participate

$$E[u^P(\mu_i, \sigma_i)] \geq E[u^I(\mu_i, \sigma_i)]$$

$$\frac{a}{n}\sum_i \mu_i - \frac{b}{n^2}(\sum_i \mu_i)^2 - \frac{b}{n^2}\sum_i \sigma_i \geq a\mu_i - b\mu_i^2 - b\sigma_i$$

$$a(\frac{1}{n}\sum_i \mu_i - \mu_{i^*}) - b(\frac{1}{n^2}(\sum_i \mu_i)^2 - \mu_{i^*}^2) - b(\frac{1}{n^2}\sum_i \sigma_i - \sigma_{i^*}) \geq 0$$

c) Total surplus amongst the students in the pool is the sum of utilities over all students. The marginal contribution of a student to the pool is the increase in the total surplus that results from her addition to the pool. Compute the marginal contribution of each student to a pool of n identical students (i.e. $\mu_i = \mu$, $\sigma_i = \sigma$ for all i).

Answer: We know from part b) that the expected value for student's utility is:

$$E[u^P(\mu_i, \sigma_i)] = \frac{a}{n}\sum_i \mu_i - \frac{b}{n^2}(\sum_i \mu_i)^2 - \frac{b}{n^2}\sum_i \sigma_i$$

Now $\mu_i = \mu$ and $\sigma_i = \sigma$ so we get:

$$E[u^P(\mu, \sigma)] = \frac{a}{n}n\mu - \frac{b}{n^2}(n\mu)^2 - \frac{b}{n^2}n\sigma$$

$$E[u^P(\mu, \sigma)] = a\mu - b\mu^2 - \frac{b}{n}\sigma$$

Let's then calculate the marginal contribution such that we first calculate how much each student gets more surplus when one student is added to the pool.

$$a\mu - b\mu^2 - \frac{b}{n+1}\sigma - (a\mu - b\mu^2 - \frac{b}{n}\sigma)$$

$$-\frac{b}{n+1}\sigma + \frac{b}{n}\sigma$$

$$(-\frac{1}{n+1} + \frac{1}{n})b\sigma$$

$$\frac{n+1-n}{n(n+1)}b\sigma$$

$$\frac{1}{n^2+n}b\sigma$$

Now we see that this is decreasing in n, which is very intuitive. The pool is used to smooth consumption, i.e. eliminate the variance of the scholarship. Think about what is the effect of adding one student to the pool of two students (increasing n from 2 to 3) compared to the effect of adding one student to the pool of 100 students (increasing n from 100 to 101).
Let's then calculate the marginal contribution as follows:

$$\frac{n}{n^2+n}b\sigma + a\mu - b\mu^2 - \frac{1}{n+1}b\sigma$$

$$a\mu - b\mu^2$$

4. A patient sister s negotiates with her impatient brother b over how to share their monthly candy portion so that the monthly shares of the two children satisfy $x_{s,t} + x_{b,t} = 1$ over the coming 12 months, $t \in \{1,2,...,12\}$. Let $1 > \beta_s > \beta_b > 0$ be the children's (monthly) discount factors. Suppose that the intertemporal utility function of child $i \in \{s,b\}$ is given by

$$U^i(x_{i,1},...,x_{i,12}) = \sum_{t=1}^{12} \beta_i^t x_{i,t}.$$

a) What are the Pareto-efficient allocations?

Answer: As the set of possible utilities is convex, we can use the Proposition 1.3. from the lecture notes. Thus, the Pareto-efficient allocations are those that solve the following weighted utilitarian maximization problem for some $(\lambda_1, \lambda_2) \neq 0$:

$$\max_{x_{s,t}} \lambda_1 \sum_{t=1}^{12} \beta_s^t x_{s,t} + \lambda_2 \sum_{t=1}^{12} \beta_b^t (1 - x_{s,t})$$

We know that the maximum of a function is either at $f' = 0$ or it lies in the boundary of the domain. From the maximization problem we get that $f' = 0$ if:

$$\lambda_1 \beta_s^t = \lambda_2 \beta_b^t$$

Meaning that in the interior solution it doesn't matter how the candy is shared. However, if

$$\lambda_1 \beta_s^t > \lambda_2 \beta_b^t$$

then the maximum lies in the boundary, i.e. $x_s = 1$ and $1 - x_s = x_b = 0$. On the other hand, if

$$\lambda_1 \beta_s^t < \lambda_2 \beta_b^t$$

then the maximum lies in the other boundary of the domain, i.e. $x_s = 0$ and $1 - x_s = x_b = 1$. Furthermore, we know that $1 > \beta_s > \beta_b > 0$, thus

- $\lambda_1 \beta_s^t > \lambda_2 \beta_b^t$ implies that $\lambda_1 \beta_s^{t'} > \lambda_2 \beta_b^{t'}$ for all $t' > t$,
- $\lambda_1 \beta_s^t < \lambda_2 \beta_b^t$ implies that $\lambda_1 \beta_s^{t'} < \lambda_2 \beta_b^{t'}$ for all $t' < t$.

Hence, Pareto-efficient allocations are such that until some t^*, the impatient brother consumes everything, and after t^*, the sister consumes everything. Thus, the Pareto set:

$$(x_s, x_b) : \begin{cases} x_{s,t} = 1 \text{ and } x_{b,t} = 0 \text{ for all } t > t^*, \\ x_{s,t} = 0 \text{ and } x_{b,t} = 1 \text{ for all } t < t^*, \\ x_{s,t} = x \text{ and } x_{b,t} = 1 - x \text{ for all } t = t^*. \end{cases}$$

where $t^* \in [0, 12]$ and $x \in [0, 1]$.

Note that t^* can be either 0 or 12 meaning that it is Pareto-optimal for the sister to have all the candy in all of the periods and for the brother to have all of the candy in all of the periods.

b) Suppose that the children have the option to ask their parents to allocate $\frac{1}{2}$ to each child in each month. What are the Pareto-efficient allocations that are acceptable to both children?

Answer: In this exercise, let's denote the discount factors such that $\beta_s = \delta$ and $\beta_b = \beta$ for clarity. Thus now, $1 > \delta > \beta > 0$. We know from part a), that in Pareto-efficient allocations the impatient brother gets the candy in the early periods, and the more patient sister in the late periods. We know that now the Pareto-efficient allocations are limited by the fact that both the sister and the brother can get half of the candy in each period. Thus,

$$\sum_{t=1}^{t^*} \beta^t \geq \sum_{t=1}^{12} \frac{1}{2} \beta^t$$

The sums here are sums of geometric sequences, thus we have:

$$\frac{1 - \beta^{t^*}}{1 - \beta} \geq \frac{1}{2} \frac{1 - \beta^{12}}{1 - \beta}$$

$$2\beta^{t^*} \leq 1+\beta^{12}$$

Let's find the t^* where the inequality binds, i.e.

$$2\beta^{t^*} = 1+\beta^{12} \tag{1}$$

We are interested in seeing how t^* behaves as a function of β. Thus we want to know the sign of $\frac{\partial t^*}{\partial \beta}$. We can find this out by using total differentiation:

$$\frac{\partial t^*}{\partial \beta} = -\frac{\frac{\partial f}{\partial \beta}}{\frac{\partial f}{\partial t^*}}$$

$$\frac{\partial t^*}{\partial \beta} = -\frac{t^* 2\beta^{t^*-1} - 12\beta^{11}}{2\beta^{t^*} ln(\beta)}$$

The denominator is negative since $0 < \beta < 1$. We are interested in the case, where $\frac{\partial t^*}{\partial \beta}$ is positive, i.e. t^* increases as β increases. Hence, the numerator can be written as follows:

$$t^* 2\beta^{t^*-1} \geq 12\beta^{11} \text{ iff } t^* 2\beta^{t^*} \geq 12\beta^{12}$$

Using equation (1), we can write this as:

$$t^*(1+\beta^{12}) \geq 12\beta^{12}$$

$$t^* \geq (12-t^*)\beta^{12} \tag{2}$$

We can write our original inequality as follows:

$$\sum_{t=1}^{t^*} \beta^t \geq \sum_{t=1}^{12} \frac{1}{2}\beta^t$$

$$\sum_{t=1}^{t^*} \beta^t \geq \sum_{t=1}^{t^*} \frac{1}{2}\beta^t + \sum_{t=t^*+1}^{12} \frac{1}{2}\beta^t$$

$$\frac{1}{2}\sum_{t=1}^{t^*} \beta^t \geq \frac{1}{2}\sum_{t=t^*+1}^{12} \beta^t$$

$$\sum_{t=1}^{t^*} \beta^t \geq \sum_{t=t^*+1}^{12} \beta^t \tag{3}$$

The left hand side of equation 2 is bigger than the left hand side of equation 3 and the right hand side of equation 2 is smaller that the right hand side of equation 3. Hence, the second term in the total differentiation equation is positive and we can conclude that t^* is increasing in β.

Let's denote the solution for the brother as $t^*(\beta)$ and for the sister as $t^*(\delta)$. Since the sister is indifferent between having half of the candy in all periods or having all of the candies from t^*+1 onwards, she is also indifferent between these alternatives and having all of the candies in the first t^* periods and none in t^*+1 onwards.

The monotonicity result above implies that the time of indifference $t^*(\delta)$ for the sister is after $t^*(\beta)$. For any integer t^k such that $t^*(\beta) \leq t^k \leq t^*(\delta)$, both siblings agree to let the brother have all the candy in the periods before t^k and the sister in periods after t^k. The siblings can share the candy in a single period t^k as long as the acceptability condition is satisfied.

c) How does your answer change if $U_i(x_{i,1},...,x_{i,12}) = \sum_{t=1}^{12} \beta_i^t \ln(x_{i,t})$?

Answer: To find the Pareto-efficient allocations, we need to solve the following maximization problem:

$$\max_{x_{s,t}} \lambda_1 \sum_{t}^{12} \beta_s^t ln x_{s,t} + \lambda_2 \sum_{t}^{12} \beta_b^t ln(1 - x_{s,t})$$

$$\text{s.t.} \begin{cases} x_{s,t} \geq 0 \\ 1 - x_{s,t} \geq 0 \end{cases}$$

The first order condition yields:

$$\frac{\lambda_1 \beta_s^t}{x_{s,t}} = \frac{\lambda_2 \beta_b^t}{1 - x_{s,t}}$$

$$\lambda_1 \beta_s^t (1 - x_{s,t}) = \lambda_2 \beta_b^t x_{s,t}$$

$$\lambda_1 \beta_s^t - \lambda_1 \beta_s^t x_{s,t} = \lambda_2 \beta_b^t x_{s,t}$$

$$\lambda_2 \beta_b^t x_{s,t} + \lambda_1 \beta_s^t x_{s,t} = \lambda_1 \beta_s^t$$

$$(\lambda_2 \beta_b^t + \lambda_1 \beta_s^t) x_{s,t} = \lambda_1 \beta_s^t$$

$$x_{s,t} = \frac{\lambda_1 \beta_s^t}{\lambda_2 \beta_b^t + \lambda_1 \beta_s^t}$$

Thus, the candy is divided according to equation above, or then we are in a corner solution where either the sister or the brother gets all of the candy in all the periods.

5. Consider housing allocations in a society.

a) Show that adding an agent and the house that she occupies may make some of the original agents worse off in the equilibrium of the new society relative to the equilibrium of the original society.

Answer: Lets consider an example where there are two agents in the original society with houses a and b. They both prefer each other's houses and in the equilibrium they change houses and are both better off. Then let's add a third agent and a house c. Now if the agent with a house a prefers house c even more than house b and the new agent with a house c prefers house a then they end up changing houses. Now the agent with a house b ends up in the same house even though they would have preferred house a instead. Thus, they are worse off than in the original equilibrium.

b) Suppose 5 unoccupied houses are located on a line and each of the 5 agents cares about the house and her nearest neighbor (or neighbors if not at the end of the line). Describe a process for finding a Pareto-efficient allocation of the houses to the agents.

Answer: Consider letting one of the agents to choose their favorite house and then their favorite neighbours. Then the neighbours can choose their favorite neighbours and so on, until each house is occupied. Let's see if this mechanism ends up in a Pareto-efficient allocation. Let's go through the agents one by one:

- Agent who chooses first: The first agent can choose their favorite house and favorite neighbor/s. There is now way we can make him better off because they have chosen their ideal allocation. Furthermore, in a Pareto-improvement we cannot make any agent worse off so the agent who chose first must remain in their favorite house with their favorite neighbor/s.
- Neighbor/s of the first agent: We cannot change their house/s because that would make the first agent worse off. Furthermore, they can choose their favorite neighbor/s so their are in their ideal allocation considering the house fixed. In other words, the only way to make a Pareto-improvement for these agents would be to change their houses or remove the first agent from their neighbors but these changes would make the first agent worse off. Thus, there are no possible Pareto-improvements.
- Neighbor/s of the second agent/s: The logic is analogous as above. We cannot change the houses of these neighbors because it would make the earlier agents worse off and they can choose their favorite neighbors from the remaining agents. This logic holds until the end of the process.

Thus, we see that this mechanism ends up in a Pareto-efficient allocation. This mechanism is a modified version of 'Serial dictatorship'.[2]

6. Consider an economy with n agents. Let X be the set of alternatives available in this economy. For each pair $(x,y) \in X \times X$, define the variable d_i for each $i \in \{1,...,n\}$ as follows:

$$d_i = \begin{cases} 1 \text{ if } x \succ y, \\ 0 \text{ if } x \sim y, \\ -1 \text{ if } y \succ x. \end{cases}$$

[2]See Proposition 8.3 and its proof from book: Osborne and Rubinstein: Models in Microeconomic Theory, 2020 on page 112.

A social choice function is a function $f : \{-1,0,1\}^n \rightarrow \{-1,0,1\}$ (with the same interpretation as above). Let $d = (d_1,...,d_n)$. The majority decision rule is defined as follows:

$$f(d_1,...,d_n) = \begin{cases} 1 \text{ if } \sum_{i=1}^n d_i > 0, \\ 0 \text{ if } \sum_{i=1}^n d_i = 0, \\ -1 \text{ if } \sum_{i=1}^n d_i < 0. \end{cases}$$

Let $n^+(d) = \#\{i \text{ such that } d_i = 1\}$ and $n_-(d) = \#\{i \text{ such that } d_i = -1\}$. A social choice function is said to be anonymous if $f(d) = f(d')$ whenever $n^+(d) = n^+(d')$ and $n_-(d) = n_-(d') =$. In other words, the rule treats all individuals in the same manner. A social choice function is neutral if $f(-d) = -f(d)$. A social choice function is responsive if $f(d) \geq 0$ and $d' > d$ imply that $f(d') = 1$.

a) Show that the majority rule is anonymous, neutral and responsive.

Answer: Let's first show anonymity: For every profile d:

$$n^+(d) - n_-(d) = \sum_{i=1}^n d_i$$

Let's then take d and d' such that $n^+(d) = n^+(d')$ and $n_-(d) = n_-(d')$. Then we have:

$$\sum_{i=1}^n d_i = n^+(d) - n_-(d) = n^+(d') - n_-(d') = \sum_{i=1}^n d'_i$$

Thus

$$\sum_{i=1}^n d_i = \sum_{i=1}^n d'_i$$
$$f(d) = f(d')$$

Second, lets show neutrality $f(-d) = -f(d)$:

$$f(-d) = \begin{cases} 1 \text{ if } \sum_{i=1}^n -d_i > 0, \\ 0 \text{ if } \sum_{i=1}^n -d_i = 0, \\ -1 \text{ if } \sum_{i=1}^n -d_i < 0. \end{cases} = \begin{cases} 1 \text{ if } \sum_{i=1}^n d_i < 0, \\ 0 \text{ if } \sum_{i=1}^n d_i = 0, \\ -1 \text{ if } \sum_{i=1}^n d_i > 0. \end{cases} = -f(d)$$

And finally let's show responsiveness:

$$f(d) \geq 0 \Rightarrow \sum_{i=1}^n d_i \geq 0$$

$$d' > d \Rightarrow \sum_{i=1}^n d'_i > \sum_{i=1}^n d_i$$

$$f(d) \geq 0 \wedge d' > d \Rightarrow \sum_{i=1}^n d'_i > \sum_{i=1}^n d_i \geq 0 \Rightarrow \sum_{i=1}^n d'_i > 0 \Rightarrow f(d') = 1$$

b) Show that whenever f is anonymous and neutral, $n^+(d) = n_-(d)$ implies that $f(d) = 0$.

Answer: Take any profile d such that $n^+(d) = n_-(d)$. Notice that we have $n^+(d) = n_-(-d)$ and $n^+(-d) = n_-(d)$. Hence, we also have $n^+(d) = n^+(-d)$ and $n_-(d) = n_-(-d)$. By anonymity:

$$f(d) = f(-d)$$

and by neutrality

$$f(-d) = -f(d)$$

Thus

$$f(d) = -f(d) \Rightarrow f(d) = 0$$

c) Prove that whenever f is anonymous, neutral and responsive, it is given by the majority rule.

Answer: In part b), we proved that whenever f is anonymous and neutral,

$$\sum_{i=1}^{N} d_i = 0 \Rightarrow f(d) = 0.$$

Take any profile d such that $\sum_{i=1}^{N} d_i > 0$. Then we can find profile d' such that $d > d'$ and $\sum_{i=1}^{N} d_i' = 0$. Based on the equation above, we know that $f(d') = 0$ and by responsiveness $f(d) = 1$. Hence, for any profile d:

$$\sum_{i=1}^{N} d_i > 0 \Rightarrow f(d) = 1.$$

Then we can take any profile d such that $\sum_{i=1}^{N} d_i < 0$. Then we can find profile d' such that $-d > -d'$ and $\sum_{i=1}^{N} -d_i' = 0$. Similarly as above, we know that $\sum_{i=1}^{N} -d_i' = 0$ implies that $f(-d') = 0$ and by responsiveness $f(-d) = 1$. Then by neutrality of f, $f(d) = -1$. Hence, for any profile d:

$$\sum_{i=1}^{N} d_i < 0 \Rightarrow f(d) = -1.$$

Now we have shown that whenever f is anonymous, neutral and responsive, it is given by the majority rule.

7. Consider the single-dimensional spatial model where the set of alternatives is given by the interval $X = [0,1]$ and there are an odd number of voters $i \in \{1, ..., n\}$. Each voter has rational preferences over X. Assume further that for each i, there is an ideal alternative $x_i^* \in [0,1]$ and that the preferences are single-peaked, i.e.

$$x < x' < x_i^* \Rightarrow x_i^* \succ x' \succ x \text{ and } x > x' > x_i^* \Rightarrow x_i^* \succ x' \succ x$$

a) Show that the societal preference induced by majority voting between pairs of alternatives is complete and transitive.

Answer: Preferences of voter $i \in \{1,...,n\}$ on X are given by $\succeq_i$. For any two alternatives $x,y \in X$, let $n(x,y)$ denote the number of voters preferring x to y, i.e. $n(x,y) = \#\{i|x \succ_i y\}$ and let $\succeq^M$ denote the societal preference induced by majority voting:

$$x \succeq^M y \Leftrightarrow n(x,y) \geq n(y,x)$$

$\succeq^M$ is a complete social preference allocation, since for any two alternatives $x,y \in [0,1]$, we have $n(x,y) \geq n(y,x)$ or $n(x,y) \leq n(y,x)$. To show transitivity, we need to show that for all $x,y,z \in X$, $x \succeq^M y$ and $y \succeq^M z$ imply $x \succeq^M z$. So let's suppose that $n(x,y) \geq n(y,x)$ and $n(y,z) \geq n(z,y)$. Then we need to show that $n(x,z) \geq n(z,x)$ is then implied.

If $x = y$ or $y = z$ or $x = z$ then the result is immediate.

So let's consider the case when $x \neq y, y \neq z, x \neq z$. There are 6 ways how x,y,z could be ordered with respect to each other on the unit interval. We may consider the cases one by one.

Case 1: $x < y < z$
By the single-peakedness of voters' preferences, we have for any voter i:

$$x \succeq_i y \Rightarrow x_i^* < y \Rightarrow y \succ_i z \Rightarrow x \succ_i z$$

So

$$n(x,z) \geq n(x,y)$$

In similar fashion, for any voter j:

$$z \succeq_j x \Rightarrow x_j^* > x \Rightarrow y \succ_j x$$

So

$$n(y,x) \geq n(z,x)$$

Now we can combine our assumptions with these results and we get:

$$n(x,z) \geq n(x,y) \geq n(y,x) \geq n(z,x)$$

Thus we have shown that

$$n(x,z) \geq n(z,x)$$

Case 2: $x < z < y$
Similar to the previous case, by the single-peakedness and rationality of voters' preferences, for any voter i:

$$x \succeq_i y \Rightarrow z \succeq_i y$$

So

$$n(z,y) \geq n(x,y)$$

Also, for any voter j:

$$y \succeq_j z \Rightarrow y \succeq_j x$$

So

$$n(y,x) \geq n(y,z)$$

Combining these findings with our assumptions $n(x,y) \geq n(y,x)$ and $n(y,z) \geq n(z,y)$ gives us:

$$n(x,y) = n(y,x) = n(y,z) = n(z,y)$$

Since the number of voters is odd, then the previous implies that there is a voter k such that $z \sim_k y$ and since they have single-peaked preferences, then $y \succ_k x$. But then necessarily $n(y,x) > n(y,z)$ which is a contradiction. Hence, $x < z < y$ is not a possible ordering under our assumptions.

Case 3: $y < x < z$ By the rationality and single-peakedness of voters' preferences, for any voter i,

$$y \succeq_i z \Rightarrow x \succeq_i z$$

So

$$n(x,z) \geq n(y,z)$$

And for any voter j

$$z \succeq_j x \Rightarrow z \succeq_j y$$

So

$$n(x,y) \geq n(z,x)$$

Combining these findings with the assumption $n(y,z) \geq n(z,y)$, gives us:

$$n(x,z) \geq n(z,x)$$

Case 4: $y < z < x$
Mirror image of $x < z < y$ so analogous arguments apply.

Case 5: $z < x < y$
Mirror image of $y < x < z$ so analogous arguments apply.

Case 6: $z < y < x$
Mirror image of $x < y < z$ so analogous arguments apply.

b) Show that the ideal point of the median voter (i.e. the median of the set $\{x_1^*, \ldots, x_n^*\}$) is strictly preferred to any other alternative in the social preference induced by majority voting.

Answer: Let m denote the median voter. Consider any $x \in X$ such that $x < x_m^*$. By the single-peakedness of voters' preferences, for every i such that $x_i^* \geq x_m^*$, $x_m^* \succ_i x$. Since $\#\{i|x_i^* \geq x_m^*\} > \frac{n}{2}$, we have $n(x_m^*, x) > n(x, x_m^*)$, that is, $x_m^* \succ^M x$.

Analogously, consider any $x' \in X$ such that $x' > x_m^*$. By the single-peakedness of voters' preferences, for every i such that $x_i^* \leq x_m^*$, $x_m^* \succ_i x'$. Since $\#\{i|x_i^* \leq x_m^*\} > \frac{n}{2}$, we have $n(x_m^*, x') > n(x', x_m^*)$, that is, $x_m^* \succ^M x'$.

c) Is the median voter a dictator in the sense of Arrow's theorem?

Answer: No. In Arrow's theorem, the dictator is an agent whose preferences determine the social preference regardless of the preference profile of the agents. But here the identity of the median voter depends on the preference profile.