Solutions: Problem Set 2

1. Consider a society where I agents share L physical goods. The goods are infinitely divisible and denote by $\bar{x}_l$ the total amount of good l available in the society. Each agent i derives a material utility x_i from consuming vector $\mathbf{x_i} = (x_{i,1},...,x_{i,L})$ of the goods. Feasible consumptions are non-negative and $\sum_i x_{i,l} \leq \bar{x}_l$ for all $l \in \{1,...,L\}$.

a) Assume that all agents i have strictly increasing material preferences in their physical consumption vectors $u_i(x_{i,1},...,x_{i,L})$. Assume that their true subjective payoffs $U_i : \mathbb{R}^I_+ \to \mathbb{R}$ are other regarding and altruistic in the sense that each $U_i(u_1(\mathbf{x_1}),...,u_L(\mathbf{x_I}))$ is a strictly increasing function. Let $\mathbf{x} = (\mathbf{x_1},...,\mathbf{x_I})$ be Pareto-efficient for the material payoff functions. Is $\mathbf{x}$ also Pareto-efficient with respect to the subjective preferences U_i?

Answer: No. We can take an example where agent j gets all of all goods, i.e. $\mathbf{x_j} = (\bar{x}_1,...,\bar{x}_L)$. This allocation is obviously Pareto-efficient for the material payoff functions. Let's then assume that the strictly increasing subjective payoff function is as follows:

$$U_i = (1-\alpha)u_i + \alpha(\frac{1}{n}\sum_{i=1}^{n} u_i)$$

where $0 < \alpha < 1$. The material payoff function is:

$$u_i(\mathbf{x_i}) = \sum_{l=1}^{L} \sqrt{x_{i,l}}$$

As $u_i(\mathbf{x_i})$ has diminishing marginal utility, agent j would like to smooth consumption between agents and give up his own consumption to other agents because the increase in the subjective utility would be higher than the decrease. Thus, $\mathbf{x}$ is not necessarily Pareto-efficient with respect to the subjective preferences U_i.

b) Let $\mathbf{x}$ be Pareto-efficient with respect to the subjective preferences. Is it also Pareto-efficient with respect to the material payoffs?

Answer: Yes. Let's prove this by contrapositive. If $\mathbf{x}$ is not Pareto-efficient with respect to material payoffs, it must be that there exists an allocation $\mathbf{x}^*$ such that $u_i(\mathbf{x_i^*}) \geq u_i(\mathbf{x_i})$ for all agents and $u_i(\mathbf{x_i^*}) > u_i(\mathbf{x_i})$ for at least one agent. However, we know that the subjective preferences U_i are strictly increasing in the vector u for all i. Hence, $u_i(\mathbf{x_i^*}) > u_i(\mathbf{x_i})$ for one agent implies $U_i(\mathbf{x}^*) > U_i(\mathbf{x})$ for all agents, which would mean that $U_i(\mathbf{x})$ is not Pareto-efficient. Thus, we can conclude that Pareto-efficiency with respect to subjective preferences implies Pareto-efficiency with respect to the material payoffs.

2. Consider an economy with five agents $N = \{1,2,3,4,5\}$ and five houses $H = \{a,b,c,d,e\}$. The individual preferences are given in the following table where columns represent the

agents and the houses are ranked in the descending order of preference within the columns. The initial allocation is represented by the boxed elements in the table.

1	2	3	4	5
b	a	d	c	a
[a]	c	a	b	[e]
c	d	[c]	[d]	c
d	[b]	e	e	d
e	e	b	a	b

a) Find a market equilibrium of this economy.

Answer: The market equilibrium is a pair of allocation AND prices. Equilibrium allocation is:

1	2	3	4	5
[b]	[a]	[d]	[c]	a
a	c	a	b	[e]
c	d	c	d	c
d	b	e	e	d
e	e	b	a	b

This is an equilibrium with prices $p(a), p(b), p(c), p(d), p(e) \in \mathbb{R}_+$ such that $p(a) = p(b) > p(e)$ and $p(c) = p(d)$.

b) Which conditions do the house prices have to satisfy in any equilibrium of this economy?

Answer: Since the agents have strict preferences over houses, the equilibrium allocation is unique. However, the equilibrium prices are not unique but they must satisfy the following conditions:

$$p(a) = p(b) > p(e) \text{ and } p(c) = p(d)$$

3. Consider a housing market where N students have been assigned a house for the past year. Assume that the rooms are single-occupancy so that we can assume that there are no externalities, and the students have preferences that depend only on the room that they have been assigned. At the end of the year, K seniors leave the college and K freshmen arrive with no existing room assigned to them. The university wants to respect its old students by letting the $N-K$ to stay in their previous rooms if they so choose. Hence we have $N-K$ students with a room and K vacant rooms and students without a room.

a) Suppose first that the old assignment of rooms to students was Pareto-efficient and consider a mechanism where only the vacant rooms are allocated to the new students while the old students keep their room. Is the resulting allocation Pareto-efficient for some allocation of the new students to vacant rooms?

Answer: Not necessarily. If some of the students $N-K$ staying in their previous rooms would prefer a vacant room and one of the new students would prefer their room, we could make a Pareto-improvement by changing the rooms of these students.

b) Consider next an assignment of the rooms using the random serial dictator method, where the students are first ordered at random (all orders equally likely) and they choose the rooms according to the order. Is the resulting allocation Pareto-efficient?

Answer: Yes. A serial dictator allocation is always Pareto-efficient. Let's prove this. Let's denote the serial dictator allocation by a and let's assume that an allocation b Pareto-dominates it. This means that $b(i) \succeq^i a(i)$ for every student $i \in N$ and $b(i) \succ^i a(i)$ for at least one student $i \in N$. Let the students be ordered such that i_1 is the student who gets to choose first and i_N is the student who gets to choose last. Then let i_r be the first student for whom $b(i_r) \succ^{i_r} a(i_r)$. Then we know that $b(i_q) = a(i_q)$ for every $q < r$, meaning that all of the students for whom $q < r$ are in the same room in allocation b and a. This implies that $b(i_r) = a(i_s)$ for some student i_s with $s > r$ and hence, it then must be that $a(i_s) \succ^{i_r} a(i_r)$. However, this contradicts the serial dictator ship method where the student r would have chosen $a(i_s)$ if $a(i_s) \succ^{i_r} a(i_r)$. For more information see Proposition 8.3 and its proof from Book: Osborne and Rubinstein: Models in Microeconomic Theory, 2020 on page 112.

c) Give next the staying students property rights to their own rooms. In other words, let them either keep their room or enter a lottery where the vacant rooms and the rooms of the participating students are allocated to the new students and participating old students according to the random serial dictator method above. Is the resulting allocation Pareto-efficient?

Answer: Consider the following example: There are three students $\{i_1, i_2, i_3\}$, and three rooms $\{r_1, r_2, r_3\}$. Student i_1 is an old student and occupies room r_1. Students i_2, i_3 are new students and rooms r_2, r_3 are vacant rooms. The following matrix gives the utilities of students over rooms:

	r_1	r_2	r_3
i_1	3	4	1
i_2	4	3	1
i_3	3	4	1

The old student, i_1, has two options. They can either stay at their current room and get a utility of 3, or they can enter the lottery. If they enter the lottery, there are 6 possible outcomes as follows:

ordering	assignment of i_1	assignment of i_2	assignment of i_3
$i_1 - i_2 - i_3$	r_2	r_1	r_3
$i_1 - i_3 - i_2$	r_2	r_3	r_1
$i_2 - i_1 - i_3$	r_2	r_1	r_3
$i_2 - i_3 - i_1$	r_3	r_1	r_2
$i_3 - i_1 - i_2$	r_1	r_3	r_2
$i_3 - i_2 - i_1$	r_3	r_1	r_2

Thus the expected utility of entering to the lottery for i_1 is:

$$\frac{1}{6}u(r_1)+\frac{3}{6}u(r_2)+\frac{2}{6}u(r_3)=\frac{3}{6}+\frac{12}{6}+\frac{2}{6}=\frac{17}{6}$$

Hence, the optimal strategy for the old student i_1 is to stay at their current room. When the old student doesn't participate in the lottery, the possible outcomes are either $(i_1, r_1; i_2, r_2; i_3, r_3)$ or $(i_1, r_1; i_2, r_3; i_3, r_2)$. However, the first one of these allocations is not Pareto-efficient because students i_1, i_2 can switch their rooms such that both are better off.[1]

Hence, this mechanism doesn't always yield a Pareto-efficient allocation nor it is (ex-post) individually rational because the old students can end up in a worse room if entering the lottery.

d) (Challenging) Suggest an alternative mechanism that respects the property rights (or squatting rights) of the old students in the sense that they are guaranteed a room that is at least as desirable as their old one and where the resulting allocation is Pareto-efficient.

Answer: The mechanism is called the static TTC (top trading cycles) rule and it works as follows: Order the students at random. In the ordering of students, assign students their favorite rooms one-at-a-time following their ordering and whenever a student demands the room of an old student, modify the ordering by inserting the old student at the top. Whenever a loop of old students forms, assign each of them the room she demands and proceed. The allocation achieved by this mechanism is both individually rational (each old student is guaranteed a room that is at least as desirable as their old one) and Pareto-efficient, because eventually, it is a serial dictatorship mechanism carried out in a particular order such that old students are not "harmed".

[1]Example taken from Abdulkadiroğlu, Atila, and Tayfun Sönmez. 1999. "House Allocation with Existing Tenants."Journal of Economic Theory 88(2): 233-60. For more information about parts c) and d), see the paper.

4. Consider a society with N (even number) agents that are to be matched in pairs. Each $n \in \{1,...,N\}$ has strict preferences over the potential match partners and all agents prefer any match to staying alone.

a) What is a matching for this society?

Answer: A matching for a society (N) is a one-to-one function from N to N which satisfies the following:

$$\mu : N \to N$$

$$\mu(n) \in N \setminus n$$

$$\mu(n) = n' \Rightarrow \mu(n') = n$$

b) Does every such society have a Pareto-efficient matching? If yes, how can you find one?

Answer: Yes. We can always use the serial dictatorship mechanism to find a Pareto-efficient matching. Here, we just need to remove the chosen agents from the set of dictators. So the mechanism would work as follows:

- Choose one agent randomly to be the first serial dictator and let them choose their favorite pair.
- Choose the second serial dictator randomly from the remaining unpaired agents and let them choose their favorite pair from the remaining unpaired agents.
- Continue until there is no one left.

Since the first dictator can choose their favorite option, it cannot be changed in a Pareto-improvement. Similarly the second serial dictator chooses their favorite option from the remaining options so only way to make them better off is if they prefer the option chosen by the previous dictator and that option cannot be removed from the previous dictator in a Pareto-improvement. Thus, the serial dictatorship mechanism always yields a Pareto-efficient matching.

c) A matching is stable if no two agents prefer forming a new match to staying in their old matches. Formalize this notion and determine if all societies have a stable matching.

Answer: A matching μ is pair-wise stable for a society (X,Y) and preference profile $(\succeq^i)_{i \in X \cup Y}$ if $y \succ^x \mu(x) \Rightarrow \mu^{-1}(y) \succ^y x$.

Let's consider a society with 4 agents and following preferences:

$$
\begin{aligned}
1: &\quad 2 \succ 3 \succ 4 \\
2: &\quad 3 \succ 4 \succ 1 \\
3: &\quad 1 \succ 2 \succ 4 \\
4: &\quad 1 \succ 2 \succ 3
\end{aligned}
$$

Each of the agents 1, 2, 3 is the most preferred partner for someone. In all possible allocation, one of them must be paired with agent 4 and the other two with each other. For $\mu(4)$, another agent will prefer her over others, and $\mu(4)$ will prefer this other member over agent 4. Thus, each allocation results in a situation where $y \succ^x \mu(x)$ and $x \succ^y \mu^{-1}(y)$ meaning there are no stable matchings for this society.

5. In a two-sided matching market with the same number of agents on both sides of the market, agents on one side of the market are ordered in an arbitrary way. They choose their matching partner in this order amongst the remaining free partners.

a) Is the resulting outcome always Pareto-efficient? What if the agents have strict preferences?

Answer: Let's first see if the serial dictatorship mechanism yields a Pareto-efficient outcome without strict preferences. Consider the following example: A society consisting of $(\{1,2,3\},\{1,2,3\})$ with the following preferences:

1:	$1 \succeq 2 \succ 3$	1:	$2 \succ 3 \succ 1$
2:	$1 \succ 2 \succ 3$	2:	$1 \succ 2 \succ 3$
3:	$1 \succ 2 \succ 3$	3:	$1 \succ 3 \succ 2$

If we use the serial dictatorship mechanism so that X's choose in order 1,2,3, we get a matching (1,1 ; 2,2 ; 3,3). However, if the first agent is indifferent between options 1 and 2, then we could make a Pareto-improvement by changing the matching such that (1,2 ; 2,1 ; 3,3). Hence, without strict preferences the outcome is not always Pareto-efficient.
Let's then consider the case with strict preferences. This is a serial dictatorship mechanism in a two-sided matching market so it must be Pareto-efficient with the same logic as in 3b).

b) Is it always stable if preferences are strict?

Answer: The serial dictatorship mechanism in a two-sided matching market doesn't always yield a stable matching no matter if the preferences are strict. Consider the following example: A society consisting of $(\{1,2,3\},\{1,2,3\})$ with the following preferences:

1:	$1 \succ 2 \succ 3$	1:	$2 \succ 3 \succ 1$
2:	$1 \succ 2 \succ 3$	2:	$3 \succ 2 \succ 1$
3:	$1 \succ 2 \succ 3$	3:	$1 \succ 3 \succ 2$

Apply the serial dictatorship algorithm in which the X 's choose Y 's in the order 1, 2, 3. This matching is unstable because $2 \in X$ and $1 \in Y$ both prefer each other to the individual with whom they are matched.[2] Thus, here we see that even though the serial dictatorship mechanism yield always a Pareto-efficient matching if preferences are strict, the Pareto-efficient matching is NOT always stable. Hence, a Pareto-efficient matching doesn't imply a stable matching in a two-sided matching market.

c) Are all stable matchings Pareto-efficient?

Answer: Let's consider the definition of a stable matching: A matching μ is pair-wise stable for a society (X,Y) and preference profile $(\succeq^i)_{i\in X\cup Y}$ if $y \succ^x \mu(x) \Rightarrow \mu^{-1}(y) \succ^y x$. Then, let's assume that a matching is stable but not Pareto-efficient. Denote the Pareto-dominating matching by μ^*. Then it must be that $\mu^* \succeq^i \mu$ for all $i \in X,Y$ and $\mu^* \succ^i \mu$ for at least one $i \in X,Y$. Let's denote the agent for whom the matching $\mu^* \succ \mu$ by x and the pair that they prefer to their own original matching by y. Then we can write that $y \succ^x \mu(x)$. By the definition of Pareto-improvement we would also have that $x \succeq^y \mu^{-1}(y)$, which contradicts the assumption of stability. Hence, stability of a matching implies Pareto-efficiency.

6. Two firms $j \in \{a,b\}$ compete over three identical workers $i \in \{1,2,3\}$. If a firm hires a single worker, it produces output worth q, if it hires two, its production is worth $Q > q$. A third worker does not add anything to the production of either firm (so that it is still Q). Firms pay non-negative wages to workers that they hire. Workers' outside option of not accepting a job is normalized to 0 so that they work in the firm that offers them the higher of the two wages and at strictly positive equal wages, each firm gets one worker. The payoff to firm j is $q - w_i^j$ if it hires worker i and $Q - w_i^j - w_{i'}^j$ if it hires i and i'. Worker i gets payoff w_i^j from working in firm j.

a) What are the Pareto efficient allocations for this economy? (I.e. the assignment of workers to firms and their wages at a Pareto-efficient allocation)?

Answer: Pareto-efficiency in this case means that no firm can get a higher revenue such that all workers get the same or higher salaries and the other firm gets same or higher revenue or the other way around, no worker can get a higher salary such that both firms get the same or higher revenue and all of the other workers get the same or higher salary. Let's note the salary of the firm hiring one worker by w^a and the firm hiring two workers by w^b.

[2]The example is taken from the book: Osborne and Rubinstein: Models in Microeconomic Theory, 2020 on page 296.

For all of the workers to choose working, all of the salaries must satisfy the following:

$$w_i^j > 0$$

If a firm hires one worker, the salary must satisfy:

$$q - w_i^a > 0$$

$$w_i^a < q$$

If a firm hires two workers, the salary must satisfy:

$$Q - w_i^b - w_{i'}^b \geq q - min\{w_i^b, w_{i'}^b\} > 0$$

$$0 < max\{w_i^b, w_{i'}^b\} \leq Q - q$$

$$0 < min\{w_i^b, w_{i'}^b\} < q$$

So Pareto-efficient allocations are such that all wages are positive ($w_i^j > 0$), firm a hires one worker and firm b hires two workers. Furthermore, the wages must satisfy the following:

$$0 < w_i^a < q$$

$$0 < max\{w_i^b, w_{i'}^b\} \leq Q - q$$

$$0 < min\{w_i^b, w_{i'}^b\} < q$$

b) Assume decreasing returns scale (i.e. $Q < 2q$) and consider stable allocations. An allocation is stable if neither firm can propose new wages such that both the firm and the workers that she gets are strictly better off under the new wages than at the old wages. What are the stable allocations if $Q < 2q$?

Answer: We know from part b) that

$$0 < w_i^a < q$$

$$0 < max\{w_i^b, w_{i'}^b\} \leq Q - q \tag{1}$$

$$0 < min\{w_i^b, w_{i'}^b\} < q$$

Furthermore, now we have to check that the firm hiring one worker cannot make a worker in the other firm such an offer that the worker would want to switch firms, thus

$$Q - w_i^a - min\{w_i^b, w_{i'}^b\} \leq q - w_i^a$$

$$min\{w_i^b, w_{i'}^b\} \geq Q - q \tag{2}$$

If we combine equations 1 and 2 we see that

$$w_i^b, w_{i'}^b = Q - q$$

Thus because the firms compete for the third worker, the worker gets the whole surplus from higher production. Furthermore, the salary from the firm that has one worker must be such that the other firm doesn't want to hire that worker instead:

$$Q - 2w_i^b \geq Q - w_i^a - w_i^b$$

$$w_i^a \geq w_i^b$$

The same applies for the other way around, the salaries from the firm that has two workers must be such that the other firm doesn't want to hire any of them instead:

$$q - w_i^a \geq q - w_i^b$$

$$w_i^b \geq w_i^a$$

So the stable allocations are such that one firm hires one worker and the other firm hires two workers and all of the salaries must be equal :

$$Q - q = w_i^b = w_i^a$$

c) What are the stable allocations if $Q > 2q$?

Answer: We know from part b) that in stable allocations the following holds

$$Q - q = w_i^b = w_i^a < q$$

However, this equation never holds if $Q > 2q$. So, there are no stable allocations as the firm hiring two workers cannot pay the workers enough to keep the other firm from stealing them.