Solutions: Problem Set 3

1. Consider the following exchange economy with two agents and three goods (real Edgeworth Box). Agent 1 has linear preferences represented by the utility function

$$u_1(x_{11}, x_{12}, x_{13}) = x_{11} + 2x_{12} + 5x_{13},$$

and agent 2 has utility function

$$u_2(x_{21}, x_{22}, x_{23}) = 3x_{21} + 3x_{22} + 7x_{23}.$$

a) Let the total resources of the three goods be given by: $\bar{x_1} = \bar{x_2} = \bar{x_3} = 3$. What are the Pareto-efficient allocations?

Answer: The agents have linear preferences, meaning that $MRS_1 \neq MRS_2$ at all points, so there cannot be interior Pareto-efficient allocations. In other words, the Pareto-efficient allocations are located either at the sides or at the corners of the box. Furthermore, the Pareto-efficient allocations are those that solve the following weighted utilitarian maximization problem for some $(\lambda_1, \lambda_2) \neq 0$:

$$\max_{x_{il}} \lambda_1(x_{11} + 2x_{12} + 5x_{13}) + \lambda_2(3x_{21} + 3x_{22} + 7x_{23})$$

Such that

$$x_{11} + x_{21} = 3$$

$$x_{12} + x_{22} = 3$$

$$x_{13} + x_{23} = 3$$

We can plug the constraints into the maximization problem and then we get the following:

$$\frac{\partial}{\partial x_{11}} : \lambda_1 = 3\lambda_2$$
$$\frac{\partial}{\partial x_{12}} : 2\lambda_1 = 3\lambda_2$$
$$\frac{\partial}{\partial x_{13}} : 5\lambda_1 = 7\lambda_2$$

Hence, the Pareto-efficient allocations are as follows:

λ_1	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}
$\lambda_1 < \frac{7}{5}\lambda_2$	0	0	0	3	3	3
$\lambda_1 = \frac{7}{5}\lambda_2$	0	0	x_{13}	3	3	$3 - x_{13}$
$\frac{7}{5}\lambda_2 < \lambda_1 < \frac{3}{2}\lambda_2$	0	0	3	3	3	0
$\lambda_1 = \frac{3}{2}\lambda_2$	0	x_{12}	3	3	$3 - x_{12}$	0
$\frac{3}{2}\lambda_2 < \lambda_1 < 3\lambda_2$	0	3	3	3	0	0
$\lambda_1 = 3\lambda_2$	x_{11}	3	3	$3 - x_{11}$	0	0
$\lambda_1 > 3\lambda_2$	3	3	3	0	0	0

b) Suppose that the initial endowments of the two agents are: $w_{11} = 2, w_{12} = 2, w_{13} = 1$ and $w_{21} = 1, w_{22} = 1, w_{23} = 2$. Compute the equilibrium prices and the equilibrium allocation for this economy.

Answer: First of all, we know by the first welfare theorem that the competitive equilibrium allocation is Pareto-efficient. Hence, we know that it must be one of the Pareto-efficient allocations that we found in part a). We can easily see from agents' utility functions that agent 2 is never going to end up in a situation where they would have none of the good 3, because even if he would get all of the other goods, his utility would be less than with original endowments:

$$3*1+3*1+7*2 = 20 > 3*3+3*3 = 18$$

Hence, we know that agent 1 can never hold all three units of good 3. Based on part a), we know that Pareto-efficient allocations are then such that agent 1 gets nothing (which is not rational in a situation where they have positive amount of initial endowments) or such that agent 1 gets only some amount of good 3 ($x_{11} = 0, x_{12} = 0, x_{13} = x$). Then we know that agent 2 has an interior consumption vector and hence optimality requires that the MRS of this agent be equal to the price ratio:

$$\frac{\frac{\partial u_2(x_{21},x_{22},x_{23})}{\partial x_{21}}}{\frac{\partial u_2(x_{21},x_{22},x_{23})}{\partial x_{22}}} = \frac{p_1}{p_2}$$

$$\frac{\frac{\partial u_2(x_{21},x_{22},x_{23})}{\partial x_{22}}}{\frac{\partial u_2(x_{21},x_{22},x_{23})}{\partial x_{23}}} = \frac{p_2}{p_3}$$

$$\frac{3}{3} = \frac{p_1}{p_2}$$

$$\frac{3}{7} = \frac{p_2}{p_3}$$

So we know that prices are $p_1 = 3\alpha, p_2 = 3\alpha, p_3 = 7\alpha$ such that α is some positive multiplier. Then we can calculate the amount of x_{13} by using agent 1's budget constraint:

$$w_{11} * p_1 + w_{12} * p_2 + w_{13} * p_3 = x_{11} * p_1 + x_{12} * p_2 + x_{13} * p_3$$

$$2*3+2*3+1*7 = x_{13}*7$$
$$x_{13} = \frac{19}{7}$$

Thus the equilibrium allocation is $(0,0,\frac{19}{7}),(3,3,\frac{2}{7})$ and prices are $p_1 = 3\alpha, p_2 = 3\alpha, p_3 = 7\alpha$ such that α is some positive multiplier.

2. Two consumers have identical Cobb-Douglas preferences for L goods $(x_{i1},...,x_{iL})$ given by:

$$u_i(x_i) = \prod_{l=1}^{L} x_{il}^{\alpha_l}$$

where $0 < \alpha_l < 1$ for all l.

a) Let ω denote the vector of total resources for the economy and find the Pareto-efficient allocations.

Answer: In an interior allocation $MRS_1 = MRS_2$ for all l. Thus we get the following equation:

$$\frac{\frac{\partial u_1(x_1)}{\partial x_{1l}}}{\frac{\partial u_1(x_1)}{\partial x_{1j}}} = \frac{\frac{\partial u_2(x_2)}{\partial x_{2l}}}{\frac{\partial u_2(x_2)}{\partial x_{2j}}}$$

$$\frac{\alpha_l x_{1j}}{\alpha_j x_{1l}} = \frac{\alpha_l x_{2j}}{\alpha_j x_{2l}}$$

If we rearrange terms we get:

$$\frac{x_{1j}}{x_{2j}} = \frac{x_{1l}}{x_{2l}}$$

Hence, we see that the shares that the agents get of each good are constant over all goods. Thus, the Pareto-efficient allocations are such that $x_{1l} = \lambda\omega_l$ and $x_{2l} = (1-\lambda)\omega_l$ such that $0 \leq \lambda \leq 1$. In addition, $\mathbf{x_1} = \lambda\omega$ and $\mathbf{x_2} = (1-\lambda)\omega$.

b) For Pareto-efficient allocations, compute the shares $(s_{21} = \frac{x_{2l}}{\omega_l})$ across the different goods l.

Answer: The share is

$$s_{21} = \frac{(1-\lambda)\omega_l}{\omega_l}$$
$$s_{21} = 1-\lambda$$

like showed in part a) and the share is constant for all $l \in L$.

3. Consider the following exchange economy.

a) In an economy, two consumers have utility functions:

$$u_1(x_{11}, x_{12}) = \ln(x_{11}) + x_{12},$$

$$u_2(x_{21},x_{22}) = \ln(x_{21}) + x_{22}.$$

Find the Pareto-efficient allocations for total resources of 2 units of good 1 and 4 units of good 2.

Answer: The Pareto-efficient allocations are those that solve the following weighted utilitarian maximization problem for some $(\lambda_1,\lambda_2) \neq 0$:

$$\max_{x_{il}} \lambda_1(\ln(x_{11}) + x_{12}) + \lambda_2(\ln(x_{21}) + x_{22})$$

Such that

$$x_{11} + x_{21} = 2,$$

$$x_{12} + x_{22} = 4.$$

We can plug the constraints into the maximization problem and then we get the following:

$$\frac{\partial}{\partial x_{11}} : \frac{\lambda_1}{x_{11}} = \frac{\lambda_2}{2 - x_{11}}$$

$$\frac{\partial}{\partial x_{12}} : \lambda_1 = \lambda_2$$

Hence, in an interior allocation it must hold:

$$x_{11} = \frac{2\lambda_1}{\lambda_1 + \lambda_1} = 1$$

$$x_{21} = 2 - \frac{2\lambda_1}{\lambda_1 + \lambda_1} = 1$$

And for x_2,

- if $\lambda_1 > \lambda_2$, then $x_{12} = 4$ and $x_{22} = 0$,
- if $\lambda_1 < \lambda_2$, then $x_{12} = 0$ and $x_{22} = 4$ and
- if $\lambda_1 = \lambda_2$, then $x_{12} = x_{12}$ and $x_{22} = 4 - x_{12}$.

Thus, the interior Pareto-efficient allocations are $(1,x_{12}),(1,4-x_{12})$. The corner Pareto-efficient allocations are $(0,0),(2,4)$ and $(2,4),(0,0)$. And the border Pareto-efficient allocations are $(x_{11},4),(2-x_{11},0)$ for $x_{11} > 1$ and $(x_{11},0),(2-x_{11},4)$ for $x_{11} < 1$.

b) Find a competitive equilibrium allocation and price for the economy where the agents have utility functions as above and the initial endowments are: $\omega_1 = (0,3), \omega_2 = (2,1)$.

Answer: We know that the equilibrium allocation is interior (because the corner or border allocations are not individually rational for the agents with these initial endowments), thus $MRS_1 = MRS_2 = \frac{p_1}{p_2}$:

$$\frac{\frac{\partial u_1(x_{11},x_{12})}{\partial x_{11}}}{\frac{\partial u_1(x_{11},x_{12})}{\partial x_{12}}} = \frac{\frac{\partial u_2(x_{21},x_{22})}{\partial x_{21}}}{\frac{\partial u_2(x_{21},x_{22})}{\partial x_{22}}} = \frac{p_1}{p_2}$$

We find the equilibrium condition

$$\frac{1}{x_{11}} = \frac{1}{x_{21}} = \frac{p_1}{p_2}$$

So we know that in the equilibrium $x_{11} = x_{21}$ and markets clear so it must be that $x_{11} = x_{21} = 1$. Then prices have to satisfy $\frac{p_1}{p_2} = 1$, thus equilibrium prices can be any prices as long as they are equal to each other. Let's then calculate the equilibrium amounts of good 2. For agent 1, it must hold that

$$p_1 x_{11} + p_2 x_{12} = p_1 \omega_1 + p_2 \omega_2$$

$$1 + x_{12} = 3$$

$$x_{12} = 2$$

Thus, the equilibrium allocation is $(1,2),(1,2)$ and $\frac{p_1}{p_2} = 1$.

c) Add a third consumer with utility function $u_3 = (x_{31}, x_{32}) = \ln(2x_{31} + x_{32})$ and $\omega_3 = (2,2)$. Find the Pareto-efficient allocations and the competitive equilibrium.

Answer: Similarly as in part b), it must be that $MRS_1 = MRS_2 = MRS_3$.

$$\frac{\frac{\partial u_1(x_{11},x_{12})}{\partial x_{11}}}{\frac{\partial u_1(x_{11},x_{12})}{\partial x_{12}}} = \frac{\frac{\partial u_2(x_{21},x_{22})}{\partial x_{21}}}{\frac{\partial u_2(x_{21},x_{22})}{\partial x_{22}}} = \frac{\frac{\partial u_3(x_{31},x_{32})}{\partial x_{31}}}{\frac{\partial u_3(x_{31},x_{32})}{\partial x_{32}}} = \frac{p_1}{p_2}$$

$$\frac{1}{x_{11}} = \frac{1}{x_{21}} = 2 = \frac{p_1}{p_2}$$

We see that $x_{11} = x_{21} = \frac{1}{2}$, $x_{31} = 3$ and prices are $p_1 = 2\lambda, p_2 = \lambda$ such that λ is some positive multiplier. We can calculate the amounts of the second good by using the budget constraints:

$$4 + 2 = 2 * 3 + 1 * x_{32}$$

$$x_{32} = 0$$

$$4 + 1 = 2 * \frac{1}{2} + 1 * x_{22}$$

$$x_{22} = 4$$

$$x_{12} = 6 - 4 - 0 = 2$$

Hence, we would get the following equilibrium allocation is $(\frac{1}{2},2),(\frac{1}{2},4),(3,0)$.

Corner Pareto-efficient allocations are $\{(0,0),(0,0),(4,6)\}$, $\{(0,0),(4,6),(0,0)\}$ and $\{(4,6),(0,0),(0,0)\}$. Border Pareto-efficient allocations are $\{(x_{11},0),(\frac{1}{2},x_{22}),(\frac{7}{2} - x_{11}, 6 - x_{22})\}$, $\{(\frac{1}{2}, x_{12}),(x_{21},0),(\frac{7}{2} - x_{21}, 6 - x_{12})\}$ and $\{(x_{11},x_{12}),(x_{11}, 6 - x_{12}),(4 - 2x_{11}, 0)\}$. Interior Pareto-efficient allocations are $\{(\frac{1}{2},x_{12}),(\frac{1}{2},x_{22}),(3,x_{32})\}$.

4. Consider an economy where all three consumers $i \in \{1,2,3\}$ have the same utility functions $u_i(x_{i1}, x_{i2}) = x_{i1}x_{i2}$, and the initial endowments of the three consumers are $\omega_1 = (1,14), \omega_2 = (1,14), \omega_3 = (27,1)$.

a) Show that the allocation $x = ((6,6),(7,7),(16,16))$ is Pareto-efficient.

Answer: Let's calculate the MRS_i:

$$MRS_i = \frac{\frac{\partial u_i(x_{i1}, x_{i2})}{\partial x_{i1}}}{\frac{\partial u_i(x_{i1}, x_{i2})}{\partial x_{i2}}}$$

$$MRS_i = \frac{x_{i2}}{x_{i1}}$$

In the proposed allocation $x_{i1} = x_{i2}$ for all i, thus $MRS_i = 1$ for all i. Hence, the proposed allocation is Pareto-efficient, since it is interior and MRS for all agents are equalized.

b) Show that this allocation is in the core of the economy.

Answer: There are seven possible blocking coalitions in this economy which are $(1),(2),(3)$, $(1,2),(1,3),(2,3),(1,2,3)$. We can check all of these one by one. It is easy and quick to check that none of the agents can block the allocation alone. Let's go through rest of the possibilities one by one:

- $(1,2,3)$: For this to be a blocking coalition, the following must hold:

  $$u(\hat{x}^1, \hat{y}^1) > u(6,6)$$

  implies

  $$\hat{x}^1 + \hat{y}^1 > 12$$

  And

  $$u(\hat{x}^2, \hat{y}^2) > u(7,7)$$

  implies

  $$\hat{x}^2 + \hat{y}^2 > 14$$

  And

  $$u(\hat{x}^3, \hat{y}^3) > u(16,16)$$

  implies

  $$\hat{x}^3 + \hat{y}^3 > 32$$

  We can combine these three:

  $$\sum_{a=1}^{3} (\hat{x}^a + \hat{y}^a) > 58$$

However, the sum of agents' initial endowments is

$$1+14+1+14+27+1=58$$

Thus, this can never be a blocking coalition.

- $(1,3)$: Similarly as above, for this pair to be a blocking coalition, the following must hold:

  $$u(\hat{x}^1,\hat{y}^1)>u(6,6)$$

  implies

  $$\hat{x}^1+\hat{y}^1>12$$

  And

  $$u(\hat{x}^3,\hat{y}^3)>u(16,16)$$

  implies

  $$\hat{x}^3+\hat{y}^3>32$$

  Thus, for the coalition to improve upon the original allocation, it must hold that

  $$\hat{x}^1+\hat{y}^1+\hat{x}^3+\hat{y}^3>44$$

  However, we know that the sum of their initial endowments is

  $$1+14+27+1=43<44$$

  Hence, this pair can never be a blocking coalition.

- (2,3): The proof above holds for this pair as agent 1 and agent 2 have the same about of initial endowments and agent 2 only has a higher utility from the original allocation.

- (1,2): For this pair to be a blocking coalition, we must have the following must hold:

  $$u(\hat{x}^1,\hat{y}^1)>u(6,6)$$

  $$u(\hat{x}^2,\hat{y}^2)>u(7,7)$$

  $$\hat{x}^1+\hat{x}^2=2$$

  $$\hat{y}^1+\hat{y}^2=28$$

  This means that also the following must hold

  $$\hat{x}^1*\hat{y}^1>36$$

  $$\hat{x}^2*\hat{y}^2>49$$

  We conclude that $\hat{x}^1\leq 2$ and $\hat{x}^2\leq 2$, which means that $\hat{y}^1>18$ and $\hat{y}^2>24,5$. Hence, $\hat{y}^1+\hat{y}^2>42,5$, which is a contradiction. Thus, this can never be a blocking coalition.

Now we have shown that the allocation is in the core of the economy.

c) Consider a replica economy where you have identical copies to the original three consumers added to the economy. Denote an allocation for this economy by $x^{(2)} = (x, x')$, where x' is the allocation for the copied consumers. Is $(((6,6),(7,7),(16,16)),((6,6),(7,7),(16,16)))$ in the core for this replica economy?

Answer: Let's see if we can find a blocking coalition for this economy. Think of the agents who get $(6,6)$. Together they have initial endowments of $(2,28)$. If they make a coalition with one of the agents who gets $(16,16)$ and has an initial endowment of $(27,1)$, they can divide their initial endowments as follows:

$$(6,4;6,4),(6,4;6,4),(16,2;16,2)$$

which is strictly better for all of the agents. Hence, the allocation $x^{(2)} = (x, x')$ is not in the core.

5. M intermediate goods $j \in \{1,...,M\}$ are produced using input vectors z_j with $z_j \in \Re^L_+$ and the production function is given by $q_j(z_j) = f_j(z_j)$ for some strictly increasing and concave function f_j. The final product q is produced from the intermediate goods according to the production function $q = min\{q_1,...,q_M\}$. Find the cost function for q in terms of the individual cost functions c_j.

Answer: The production function of the final product q is a Leontief production function, which means that the inputs must be used in exactly the right proportions or the excess is wasted. In this case it means that to produce q final products, q intermediate goods of each type must be used. Thus, the cost function for the final product q is:

$$C(c_j) = \sum_{j=1}^{M} c_j$$

6. Consider a production economy with a fixed size of available land L. All agents in the economy either work or enjoy leisure. Total amount of time available is $T \leq 2L$. Working hours are divided between cultivating barley b, denoted by t_b or cultivating rye r denoted by t_r so that leisure amounts to $T - t_b - t_r$. Land is also divided amongst barley and rye into l_b and l_r.

a) Suppose that all agents have the same preferences given by

$$u^i(b^i, r^i, t^i_b, t^i_r) = b^i r^i (T - t^i_b - t^i_r)$$

and the production functions for firms producing b and r are Leontieff:

$$b = min\{\frac{1}{2}t_b, l_b\}, r = min\{t_r, l_r\}$$

where t_j denotes the aggregate time spent cultivating j. Show that if a competitive solution to the firms' problems exists, the firms make zero profit.

Answer: First of all, we notice that the size of available land is never a constraint for the firms, thus the price of land is 0. Then, we see that agent's don't have any particular preference to cultivate barley or rye, so they will choose the one which has a higher salary. Therefore, for firms to have workers for both products, salaries must be equal:

$$w_r = w_b = w$$

Firm's maximization problem can be written as follows:

$$\max_{b,r,t_b,t_r} p_b b + p_r r - w(t_b + t_r)$$

such that

$$b = \frac{1}{2}t_b$$

$$r = t_r$$

We can plug the constraints into the maximization problem and we get the following FOC:

$$\frac{1}{2}p_b - w = 0$$

$$p_r - w = 0$$

Where we can solve for

$$\frac{1}{2}p_b = p_r = w$$

Hence, we see that if a competitive solution to the firm's problems exists, it must be that the firms make zero profit in equilibrium because

$$wt_b + wt_r - wt_b - wt_r = 0$$

b) Define a competitive equilibrium for this economy.

Answer: A competitive equilibrium of the economy is a positive price vector (p_b, p_r, w_b, w_r) and allocation $(b^i, r^i, t_b^i, t_r^i, b, r, t_b, t_r)$ such that

- for each $i \in I$, b^i, r^i, t_b^i, t_r^i is optimal considering the consumer's maximization problem and budget constraint

- for each firm b, r, t_b, t_r is optimal considering the firm's maximization problem
- market clears

$$\sum b^i = b$$
$$\sum r^i = r$$

c) Show that in any competitive equilibrium, all agents work the same total hours.

Answer: We can denote the time agent spends working by t^i and leisure by $T - t^i$. Then we can write the agent's maximization problem as follows:

$$\max_{b^i, r^i, t^i} b^i r^i (T - t^i)$$

such that

$$p_b b^i + p_r r^i = w t^i$$

FOC:

$$r^i (T - t^i) = -\lambda p_b \tag{1}$$
$$b^i (T - t^i) = -\lambda p_r \tag{2}$$
$$b^i r^i = -\lambda w \tag{3}$$
$$p_b b^i + p_r r^i - w t^i = 0 \tag{4}$$

From (4) we can solve for t^i:

$$t^i = \frac{p_b b^i + p_r r^i}{w} \tag{5}$$

Then to find an expression for r^i, we can divide (1) by (2) and we get:

$$\frac{r^i}{b^i} = \frac{p_b}{p_r}$$
$$r^i = \frac{b^i p_b}{p_r} \tag{6}$$

Now we can plug (6) into (5):

$$t^i = \frac{2 p_b b^i}{w} \tag{7}$$

Then, to find an expression for b^i, we can solve (3) for $-\lambda$:

$$-\lambda = \frac{b^i r^i}{w}$$

And then plug it into (2):

$$b^i (T - t^i) = \frac{b^i r^i p_r}{w}$$

Let's then use our expression for r^i from (6) to get:

$$b^i(T-t^i) = \frac{(b^i)^2 p_b}{w}$$

And then solve for b^i:

$$b^i = \frac{w(T-t^i)}{p_b} \tag{8}$$

Finally, we can plug this into (7):

$$t^i = 2(T-t^i)$$

$$t^i = \frac{2}{3}T$$

Now we have shown that all agents work the same total hours.

d) Solve for the competitive equilibrium prices and allocation.

Answer: In part c) we have solved for $t^i = \frac{2}{3}T$. Let's continue from there. From the firm's maximization problem (part a)) we know that $\frac{1}{2}p_b = p_r = w$. Let's plug $p_r = w$ into (2) and we get the following:

$$b^i(T-t^i) = b^i r^i$$

$$T - t^i = r^i$$

$$r^i = \frac{1}{3}T$$

Then to solve for b^i, we can use (1) and $\frac{1}{2}p_b = w$ to get:

$$r^i(T-t^i) = 2b^i r^i$$

$$b^i = \frac{T-t^i}{2}$$

$$b^i = \frac{1}{6}T$$

So we know that consumers' allocation is $(b^i, r^i, t^i) = (\frac{1}{6}T, \frac{1}{3}T, \frac{2}{3}T)$. Prices are positive and satisfy:

$$\frac{1}{2}p_b = p_r = w$$

and markets clear:

$$\sum b^i = \sum \frac{1}{6}T = b = \frac{1}{2}t_b,$$

$$\sum r^i = \sum \frac{1}{3}T = r = t_r.$$