Solutions: Problem Set 4

1. Answer the following short questions.

a) Two agents have preferences represented by utility functions $u_1(x)$ and $u_2(x)$ over allocations x in a finite set X. Assume that x^* is Pareto-efficient if an allocation is chosen once. Suppose that an $x^j \in X$ is chosen separately in two periods $j \in \{1,2\}$ and the utility function of agent i for (x_1,x_2) is given by $u_i(x^1)+u_i(x^2)$ (i.e. the utility is the sum of the utilities in each period). Prove or disprove that the allocation $x^1 = x^2 = x^*$ is Pareto-efficient in the two-period allocation problem?

Answer: $x^1 = x^2 = x^*$ is NOT Pareto-efficient in the two-period allocation problem. Consider sharing two indivisible items between two agents. Let $x \in \{0,1,2\}$ denote the number of items to 1 and therefore 2 gets $2-x$. Let $u_1(x) = x^2$ and $u_2(x) = (2-x)^2$. Then all $x \in \{0,1,2\}$ are Pareto-efficient in a single allocation problem. In the two-period problem, $(1,1)$ is Pareto-dominated by $(2,0)$ and $(0,2)$.

b) The following table lists the preferences of four agents $i \in \{1,2,3,4\}$ represented by the columns over four houses $j \in \{a,b,c,d\}$ represented by the rows. If a house j is on a higher row than j' in column i, then $j \succ_i j'$.

1	2	3	4
a	a	c	b
c	d	$\boxed{\text{a}}$	$\boxed{\text{c}}$
$\boxed{\text{d}}$	$\boxed{\text{b}}$	b	d
b	c	d	a

The initial allocation of houses in this economy is (d,b,a,c) (where agent i gets the i^{th} element in the vector as her house)? Define a competitive equilibrium for this economy and find a competitive equilibrium price and a competitive equilibrium allocation of houses.

Answer: A competitive equilibrium is an allocation and a price vector such that all agents choose their most preferred house in their budget set (price less than or equal to the price of initial endowment) and markets clear. We find the equilibrium allocation by top trading cycle (2,3,4). Equilibrium allocation is:

1	2	3	4
a	$\boxed{\text{a}}$	$\boxed{\text{c}}$	$\boxed{\text{b}}$
c	d	a	c
$\boxed{\text{d}}$	b	b	d
b	c	d	a

This is an equilibrium with prices $p(a), p(b), p(c), p(d) \in \mathbb{R}_+$ such that $p(a) = p(b) = p(c) > p(d)$.

2. Answer the following problems for an exchange economy.

a) State and prove the first welfare theorem for exchange economies.

Answer: First Welfare Theorem for Exchange Economies: Suppose all agents in an exchange economy $(N, \{\succeq_i\}_{i\in N}, \{\omega_i\}_{i\in N})$ have locally nonsatiated preferences. Then all competitive equilibrium allocations of the economy are Pareto-efficient.

Proof: Let $(\mathbf{p}, \mathbf{x})$ be a competitive equilibrium and $\mathbf{y}$ an allocation that Pareto-dominates $\mathbf{x}$. By local non-satiation, $\mathbf{p} \cdot y_i \geq \mathbf{p} \cdot x_i$ for all i and $\mathbf{p} \cdot y_i > \mathbf{p} \cdot x_i$ for some i. Summing over all i, we get $\sum_{i=1}^n \mathbf{p} \cdot y_i > \sum_{i=1}^n \mathbf{p} \cdot x_i = \sum_{i=1}^n \mathbf{p} \cdot \omega_i$, where the last equality follows from Walras' law. Hence, there must be an l such that $\sum_{i=1}^n y_{il} > \sum_{i=1}^n \omega_{il}$ showing that $\mathbf{y}$ is not feasible.

b) Find the Pareto-efficient allocations for a society consisting of two agents $i \in \{1, 2\}$ with utility functions

$$u_1(x_{11}, x_{12}) = \ln(x_{11}) + \alpha x_{12}$$

$$u_2(x_{21}, x_{22}) = \beta x_{21} + \ln(x_{22}).$$

where $\alpha, \beta > 0$, and the total quantities of the goods are $\bar{x_1} = 5$, $\bar{x_2} = 3$.

Answer: We can find the Pareto-efficient interior allocations by $MRS_1 = MRS_2$:

$$\frac{\frac{\partial u_1(x_{11}, x_{12})}{\partial x_{11}}}{\frac{\partial u_1(x_{11}, x_{12})}{\partial x_{12}}} = \frac{\frac{\partial u_2(x_{21}, x_{22})}{\partial x_{21}}}{\frac{\partial u_2(x_{21}, x_{22})}{\partial x_{22}}}$$

$$\frac{1}{\alpha\beta} = x_{11}x_{22}$$

Resource constraint $x_{11} = 5 - x_{21}$ so we get:

$$x_{12} = 3 - \frac{1}{\alpha\beta x_{11}}$$

$$x_{21} = 5 - x_{11}$$

$$x_{22} = 3 - x_{12}$$

Then we can calculate the Pareto-efficient boundary allocations which are $\{(x_{11}, 0), (5 - x_{11}, 3)\}$ when $x_{11} \leq \frac{1}{3\alpha\beta}$ and $\{(5, x_{12}), (0, 3 - x_{12})\}$ when $x_{12} \geq 3 - \frac{1}{5\alpha\beta}$. Finally, we also know that corner solutions $\{(0,0), (5,3)\}$ and $\{(5,3), (0,0)\}$ are also Pareto-efficient.

c) Find a competitive equilibrium for an economy where the agents have utility functions as above with $\alpha = \beta = 1$, and their initial endowments are: $\omega_1 = (1,2), \omega_2 = (4,1)$.

Answer: If we compare the initial endowments to the conditions for interior Pareto-efficient allocations we notice that the initial allocation is Pareto-efficient. Since preferences are convex, second welfare theorem implies that for some prices, the initial allocation is a competitive equilibrium allocation. Since the allocation is interior, we know that it must hold that:

$$MRS_1 = MRS_2 = \frac{p_1}{p_2}$$
$$1 = x_{11}x_{22} = \frac{p_1}{p_2}$$

Thus, in equilibrium the prices must be equal to each other.

3. Consider the following economy: There are two periods, two states in the second period, and two consumers. There is one physical commodity. The endowment in period 0 is 2 for both consumers. Consumer 1 has an endowment $\omega_{1s} = 1$ in both states in period 1 and consumer 2 has an endowment $\omega_{2s} = 2$ in both states in period 1. By x_0 we denote the consumption in period 0. By x_{1s} we denote the consumption in period 1 in state s. The utility functions of the consumers are given by:

$$u^i(x_0, x_{11}, x_{12}) = \ln x_0 + \sum_{s=1}^{2} p_s \ln x_{1s}.$$

a) Suppose that the agents can save their period 0 endowment so that if i saves y_i in period 0, she can consume $\omega_{is} + y_i$ in state s in period 1. Find the optimal savings for the two consumers.

Answer: As agent saves y_i in period 0, then the consumption in period 0 is:

$$2 - y_i.$$

Consumption in both states for agent 1 in period 1 is:

$$1 + y_1$$

And for agent 2:

$$2 + y_2$$

The agents maximization problem is as follows:

$$\max_{y_i} \ln(\omega_{i0} - y_i) + p_1 \ln(\omega_{i1} + y_i) + p_2 \ln(\omega_{i2} + y_i)$$

As both of the agents get the same endowment in both of the states, we can write:

$$\max_{y_i} \ln(\omega_{i0} - y_i) + \ln(\omega_{i1} + y_i)$$

FOC:

$$\frac{1}{y_i + \omega_{i1}} = \frac{1}{\omega_{i0} - y_i}$$

Hence,

$$y_i + \omega_{i1} = \omega_{i0} - y_i$$
$$y_i = \frac{\omega_{i0} - \omega_{i1}}{2}.$$

Thus, the optimal savings for the agents are:

$$y_1 = \frac{1}{2}$$
$$y_2 = 0.$$

b) Assume that the good is perishable and cannot be stored between periods. The agents have access to an asset that pays 1 in both states in period 1 and costs q in period 0. The asset is in zero net supply so that market clearing requires $z_1 + z_2 = 0$, where z_i is the asset demand of agent i. If $z_i < 0$, then i receives qz_i in period 0 and pays z_i in both states in period 1. Define a competitive equilibrium for this model and solve the equilibrium asset price q and the equilibrium asset demand and consumption allocation.

Answer: A competitive equilibrium is a positive price q and asset holdings (z_1, z_2) such that

- at price q, z_1 maximizes agent 1's problem and z_2 maximizes agent 2's problem
- markets clear: $z_1 + z_2 = 0$.

The agents face the following maximization problem:

$$\max_{z_i} \ln(\omega_{i0} - qz_i) + \ln(\omega_{i1} + z_i)$$

such that market clears:

$$z_1 + z_2 = 0$$

FOC:

$$\frac{1}{z_i + \omega_{i1}} = \frac{q}{\omega_{i0} - qz_i}$$

Hence,

$$z_i = \frac{\omega_{i0} - q\omega_{i1}}{2q}$$

Let's then plug this into the market clearing condition and we get:

$$\frac{2-q}{2q} + \frac{2-2q}{2q} = 0$$

$$q = \frac{4}{3}$$

$$z_1 = \frac{1}{4}$$

$$z_2 = -\frac{1}{4}$$

c) Continue assuming that the good is perishable and solve for the competitive equilibrium when the only asset available to the agents pays 1 unit in state 1 and nothing in state 2.

Answer: Now the agents face the following maximization problem:

$$\max_{z_i} \ln(\omega_{i0} - qz_i) + p_1 \ln(\omega_{i1} + z_i) + p_2 \ln(\omega i2)$$

Such that market clears

$$z_1 + z_2 = 0$$

FOC:

$$\frac{p_1}{\omega_{i1} + z_i} = \frac{q}{\omega_{i0} - qz_i}$$

Hence,

$$p_1(\omega_{i0} - qz_i) = q(\omega_{i1} + z_i)$$

$$p_1\omega_{i0} - q\omega_{i1} = (q + p_1 q)z_i$$

$$z_i = \frac{p_1\omega_{i0} - q\omega_{i1}}{q + p_1 q}$$

From the market clearing condition we get:

$$\frac{2p_1 - q}{q + p_1 q} + \frac{2p_1 - 2q}{q + p_1 q} = 0$$

$$q = \frac{4p_1}{3}$$

$$z_1 = \frac{1}{2(p_1 + 1)}$$

$$z_2 = -\frac{1}{2(p_1 + 1)}$$

4. An economy consists of a continuum of mass 2 of agents and a continuum of mass 3 of houses. The agents' willingness to pay for housing quality v_i is uniformly distributed on $[0,2]$ so that the utility for an agent of type v_i from owning a house of quality q_j is $v_i q_j$. The houses have a uniform quality distribution on $[0,3]$.

a) The houses are owned by absentee landlords and their opportunity cost of renting (outside option for holding the house of quality q_j) is $\gamma + \alpha q_j$. Assume that the agents and the house owners have quasilinear utilities in utility from housing and money. Find the efficient allocation of houses (determine also which houses stay with the absentee landlords).

Answer: An allocation determines

- A subset of agents' valuations $V \subseteq [0,2]$ such that agent i is assigned a house if and only if $v_i \in V$
- function $q : V \to [0,3]$ that maps agents' valuations to housing qualities

Feasibility of an allocation requires that no subset of V can have a greater mass than the set of housing qualities mapped to it. Since the utilities are quasilinear, an efficient allocation maximizes the total surplus from the housing assignment. The surplus from matching WTP v to house quality q is $vq - (\gamma + \alpha q)$. Thus, an efficient allocation solves the following maximization problem:

$$\max_{V, q(v)} \int_{v \in V} (v - \alpha) q(v) - \gamma \,\mathrm{dv}$$

Hence, in the efficient allocation, $V = [v^*, 2] \cap [0,2]$, where v^* solves

$$(v^* - \alpha)(v^* + 1) - \gamma = 0$$

That is, all the individuals with valuations greater than v^* are assigned a house. Moreover, in the efficient allocation, $q(v) = v + 1$. That is, among the agents that are assigned houses, the allocation is positive assortative so that $q(v) = v + 1$.

b) Assume that $\gamma = \alpha = 1$ and solve for the competitive equilibrium price (function) $p(q)$ for the houses and describe the equilibrium allocation of houses.

Answer: The equilibrium allocation must be Pareto-efficient, so based on part a), each agent i with value $v_i \in [\sqrt{2}, 2]$ is assigned a house of quality $v_i + 1$. Houses with qualities $q < \sqrt{2} + 1$ are left unoccupied.
The equilibrium prices must then be such that the landlords' optimal choice is to rent the houses with qualities on $[\sqrt{2} + 1, 3]$ and not rent the houses with qualities on $[0, \sqrt{2} + 1)$, so, $p(q) \geq 1 + q$ for all $q \in [\sqrt{2} + 1, 3]$ and $p(q) \leq 1 + q$ for all $q \in [0, \sqrt{2} + 1)$.
Each agent must also make their optimal choice in the equilibrium: Each i must prefer renting her equilibrium house to not renting: $p(q) \leq (q - 1)q$ for all $q \in [\sqrt{2} + 1, 3]$. v_i must be the optimal housing quality for i, so it must solve the maximization problem

$$\max_{q \in [\sqrt{2}+1, 3]} v_i q - p(q)$$

Assuming differentiability of $p(q)$, the first-order condition is $p'(q) = v_i$. Hence, the equilibrium prices must satisfy $p'(q) = q - 1$ for all $q \in [\sqrt{2} + 1, 3]$.

The following price function satisfies the conditions:

$$p(q) = \frac{1}{2}q^2 - q + \frac{3+2\sqrt{2}}{2}$$

5. Consider a large population of identical children. A child's preferences can be represented by a utility function $U : \mathbb{R} \to \mathbb{R}$, which takes a number of gifts received as input. Furthermore, $U' > 0$ and $U'' < 0$.
Christmas is coming. There are two possible states of Christmas. With a probability of $\frac{1}{2}$, Christmas is happy, in which case each child receives two Christmas gifts from Santa Claus. With the remaining probability of $\frac{1}{2}$, Christmas is miserable, in which case each child receives only one Christmas gift from Santa Claus. The state of Christmas is realized on Christmas Eve.
Children can trade three different assets. A child who holds one risk-free asset on Christmas Eve is entitled to one additional Christmas gift regardless of the state of Christmas. A holder of a procyclical asset is entitled to 2 additional Christmas gifts if Christmas is happy, and to 0 additional gifts if Christmas is miserable. A child who holds a countercyclical asset on Christmas Eve is entitled to 2 additional Christmas gifts if Christmas is miserable, and to 0 additional gifts if Christmas is happy. Denote the pre-Christmas prices of these assets by P^{RF}, P^{PC}, and P^{CC}, respectively (the prices are measured in terms of gifts). Short sales are allowed, and the assets must be in zero net supply.
Hint: Notice that in the equilibrium, there is no asset trade, and each child is indifferent between buying and selling any of the assets before Christmas.

a) Determine the following three price ratios: $P^{RF}/P^{PC}, P^{PC}/P^{CC}$, and P^{RF}/P^{CC}. (The answer should be in terms of $U'(1)$ and $U'(2)$.) Which one of the assets is the most expensive one, and which asset is the cheapest one? Which asset has the highest expected rate of return?

Answer: Denote the gift endowment from Santa Claus by c and the "gift dividends" given by asset i by d_i. The Euler equations tell us:

$$\frac{E[U'(c)d_{RF}]}{P^{RF}} = \frac{E[U'(c)d_{PC}]}{P^{PC}} = \frac{E[U'(c)d_{CC}]}{P^{CC}}$$

Let's then calculate the expected values:

$$E[U'(c)d_{RF}] = \frac{1}{2}U'(2) + \frac{1}{2}U'(1)$$

$$E[U'(c)d_{PC}] = \frac{1}{2}U'(2) * 2 = U'(2)$$

$$E[U'(c)d_{CC}] = \frac{1}{2}U'(1) * 2 = U'(1)$$

Then we can calculate the equilibrium asset price ratios:

$$P^{RF}/P^{PC} = \frac{E[U'(c)d_{RF}]}{E[U'(c)d_{PC}]}$$

$$P^{RF}/P^{PC} = \frac{\frac{1}{2}U'(2) + \frac{1}{2}U'(1)}{U'(2)}$$

$$P^{PC}/P^{CC} = \frac{E[U'(c)d_{PC}]}{E[U'(c)d_{CC}]}$$

$$P^{PC}/P^{CC} = \frac{U'(2)}{U'(1)}$$

$$P^{RF}/P^{CC} = \frac{E[U'(c)d_{RF}]}{E[U'(c)d_{CC}]}$$

$$P^{RF}/P^{CC} = \frac{\frac{1}{2}U'(2) + \frac{1}{2}U'(1)}{U'(1)}$$

Since $U'' < 0$, we know that $U'(1) > U'(2)$, so we have:

$$P^{CC} > P^{RF} > P^{PC}$$

Lower asset prices associated with higher expected rates of return.

b) Consider an otherwise similar situation, but with a small difference regarding the miserable state of Christmas: if Christmas is miserable, half of the children receive 0 Christmas gifts from Santa Claus, whereas the remaining half of the children receive 2 gifts from Santa Claus. The children are still identical ex ante. Determine the price ratio P^{PC}/P^{CC} in this situation. If $U''' = 0$ (in the relevant part of the domain), is P^{PC}/P^{CC} equal, greater, or less than in Part (a)? If $U''' > 0$, is P^{PC}/P^{CC} equal, greater,or less than in Part (a)?

Answer: Similarly as in part a), the Euler equations tell us:

$$\frac{E[U'(c)d_{PC}]}{P^{PC}} = \frac{E[U'(c)d_{CC}]}{P^{CC}}$$

Let's then calculate the expected values (which are now different compared to part a)):

$$E[U'(c)d_{PC}] = \frac{1}{2}U'(2) * 2 = U'(2)$$

$$E[U'(c)d_{CC}] = \frac{1}{2}U'(2) * 0 + \frac{1}{4}U'(0) * 2 + \frac{1}{4}U'(2) * 2 = \frac{1}{2}U'(0) + \frac{1}{2}U'(2)$$

Then we can calculate the equilibrium asset price ratios:

$$P^{PC}/P^{CC} = \frac{E[U'(c)d_{PC}]}{E[U'(c)d_{CC}]}$$

$$P^{PC}/P^{CC} = \frac{U'(2)}{\frac{1}{2}U'(0) + \frac{1}{2}U'(2)}$$

In part a) we had:

$$P^{PC}/P^{CC} = \frac{U'(2)}{U'(1)}$$

If $U''' = 0$, then

$$U'(\frac{1}{2}*0+\frac{1}{2}*2) = \frac{1}{2}U'(0)+\frac{1}{2}U'(2)$$

meaning P^{PC}/P^{CC} is equal in parts a) and b). If $U''' > 0$, then

$$U'(\frac{1}{2}*0+\frac{1}{2}*2) < \frac{1}{2}U'(0)+\frac{1}{2}U'(2)$$

meaning P^{PC}/P^{CC} is less in part b).