ELEC-E8116 Model-based control systems / exercises with solutions 7

1. Let the weight of the sensitivity function be given as

$$\frac{1}{W_s} = A\frac{\frac{s}{A\omega_0}+1}{\frac{s}{B\omega_0}+1}, \quad 0 < A << 1, B >> 1$$

Sketch a schema for the magnitude plot of the frequency response and investigate its characteristics. What is the slope in the increasing part of the curve? What is the magnitude at frequency ω_0?

Generate a second order model, where the slope is twice as large as in the previous case. Investigate again the characteristics. What is the magnitude at frequency ω_0?

Solution:

The goal is to parametrize the given weight

$$\frac{1}{W_s(j\omega)} = A\frac{\frac{j\omega}{A\omega_0}+1}{\frac{j\omega}{B\omega_0}+1} \Bigg| \cdot \frac{1}{j\omega} \quad \Rightarrow \frac{1}{W_s(j\omega)} = A\frac{\frac{1}{A\omega_0}+\frac{1}{j\omega}}{\frac{1}{B\omega_0}+\frac{1}{j\omega}}$$

Alternatively, we can have

$$\frac{1}{W_s(j\omega)} = A\frac{\frac{j\omega}{A\omega_0}+\frac{A\omega_0}{A\omega_0}}{\frac{j\omega}{B\omega_0}+\frac{B\omega_0}{B\omega_0}} = A\frac{A\omega_0 + j\omega}{B\omega_0 + j\omega}\frac{B\omega_0}{A\omega_0} = B\frac{A\omega_0 + j\omega}{B\omega_0 + j\omega}$$

Clearly at low frequencies $\frac{1}{W_s(j0)} = A$ and at high frequencies $\frac{1}{W_s(j\infty)} = B$

$$\left|\frac{1}{W_s(j\omega)}\right| = A\sqrt{\frac{1+\left(\frac{\omega}{A\omega_0}\right)^2}{1+\left(\frac{\omega}{B\omega_0}\right)^2}}$$

For $\omega \to \omega_0$,

$$\left|\frac{1}{W_s(j\omega)}\right|_{\omega=\omega_0} = A\sqrt{\frac{1+\left(\frac{1}{A}\right)^2}{1+\left(\frac{1}{B}\right)^2}} = \sqrt{\frac{A^2\left(1+\left(\frac{1}{A}\right)^2\right)}{1+\left(\frac{1}{B}\right)^2}} = \sqrt{\frac{1+A^2}{1+\frac{1}{B^2}}} \approx 1$$

because B is "large" and A is "small".

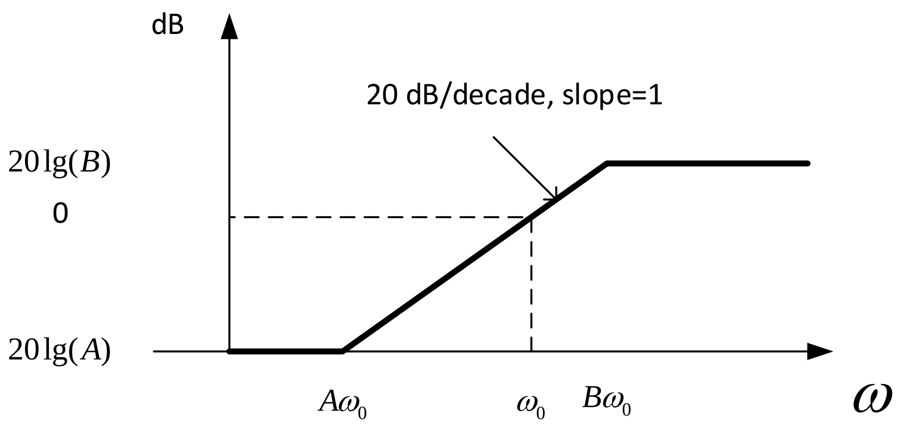
The Bode diagram (amplitude) is shown below:

Note that for the absolute value of the term $1+j\omega T$ in the frequency response it holds

$\sqrt{1+(\omega T)^2}\underset{\omega=1/T}{=}\sqrt{2}\approx 3\text{dB}$ which can be approximated as 0 dB. For higher frequencies

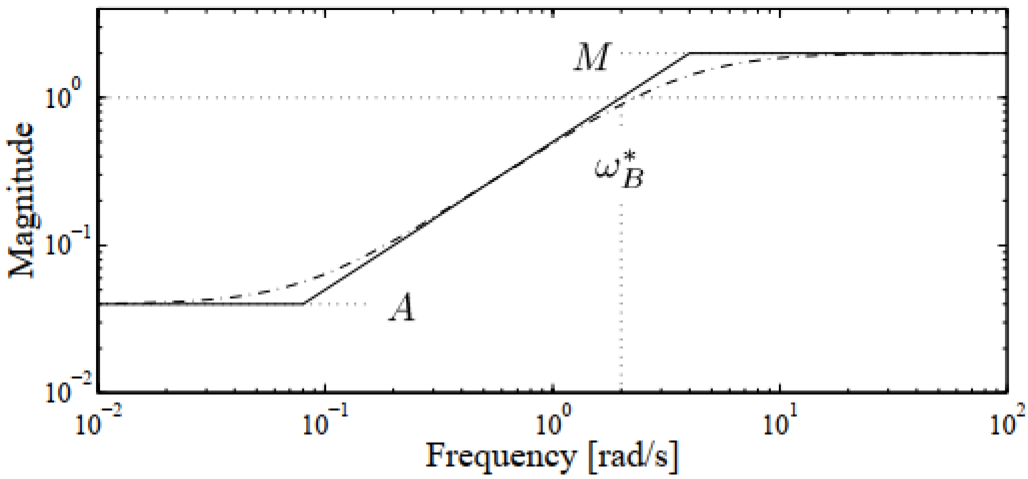
$$\sqrt{1+(\omega T)^2}\approx\sqrt{(\omega T)^2}=\omega T \Rightarrow 20\lg(\omega T)=20\lg(\omega)+20\lg(T)$$

increases 20dB/decade (slope = 1) from zero decibels at $\omega = 1/T$.



Note that in the lecture slides an example of *Mixed Sensitivity Design* was shown with the desired sensitivity weight

$$\frac{1}{W_s(s)} = \frac{s + \omega_B^* A}{\frac{s}{M} + \omega_B^*}.$$ This is the same parameterization as in the problem, by $M = B, \omega_B^* = \omega_0$.



Inverse of performance weight. Exact and asymptotic plot of $1/W_S(j\omega)$

The second order model is

$$\frac{1}{W_s} = A\frac{(\frac{j\omega}{A^{1/2}\omega_0}+1)^2}{(\frac{j\omega}{B^{1/2}\omega_0}+1)^2}$$

Similar calculus as above shows that the amplitude curve is as in the above figure but with the angular frequencies $\left(A^{1/2}\omega_0, \omega_0, B^{1/2}\omega_0\right)$ instead of $\left(A\omega_0, \omega_0, B\omega_0\right)$. The curve increases 40 dB/decade, slope is 2. Note that this is again the same as

$$\frac{1}{W_s(s)} = \frac{(s + \omega_B^* A^{1/2})^2}{(\frac{s}{M^{1/2}} + \omega_B^*)^2}.$$

2. Consider the angular frequencies $\omega_B, \omega_c, \omega_{BT}$ which are used to define the bandwidth of a controlled system. State the definitions. Prove that when the phase margin is less than 90 degrees ($PM < \pi/2$) it holds $\omega_B < \omega_c < \omega_{BT}$. Interpretations?

Solution: Definitions:

ω_B : where S crosses $1/\sqrt{2} \approx -3\text{dB}$ from below.

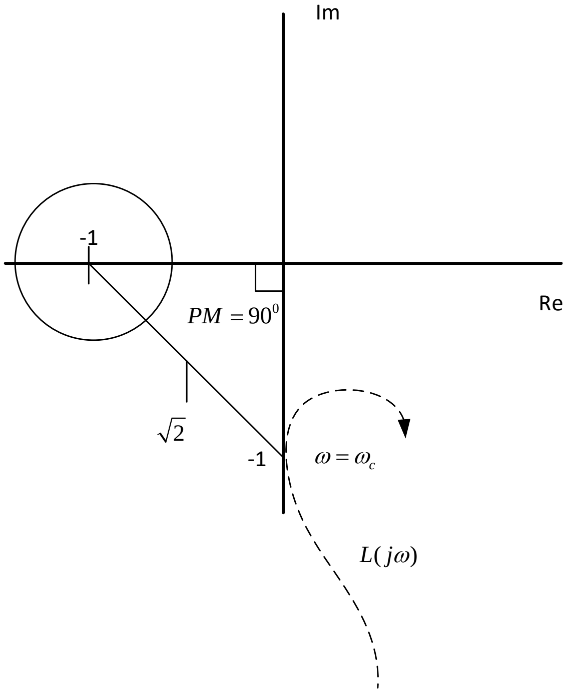
ω_c : where L crosses 1 = 0 dB (gain crossover (angular) frequency)

ω_{BT} : where T crosses $1/\sqrt{2} \approx -3\text{dB}$ from above.

At the gain crossover frequency it holds

$$|L(j\omega_c)| = 1 \Rightarrow |T(j\omega_c)| = \left|\frac{L(j\omega_c)}{1+L(j\omega_c)}\right| = \frac{|L(j\omega_c)|}{|1+L(j\omega_c)|} = \frac{1}{|1+L(j\omega_c)|} = \left|\frac{1}{1+L(j\omega_c)}\right| = |S(j\omega_c|$$

(Note that $L(j\omega_c)$ is a complex number and so $|1+L(j\omega_c)| \neq 1+|L(j\omega_c)|$.
$|1+x+jy| = \sqrt{(1+x)^2+y^2} \neq 1+\sqrt{x^2+y^2}$, except in some rare exceptional cases (when?)).



The figure shows the Nyquist diagram of L where the phase margin PM = 90 degrees. In the gain crossover frequency then
$|S(j\omega_c| = |T(j\omega_c| = 1/\sqrt{2} \approx -3\text{ dB}$ (The distance from the point (-1,0) is inversely proportional to the absolute value of S. See lecture slides, Chapter 3).

So, at ω_c all the bandwidths would coincide.

But when PM < 90 degrees $\frac{1}{|S(j\omega_c|} < \sqrt{(2)} \Rightarrow |S(j\omega_c| = |T(j\omega_c| > 1/\sqrt{2}$,

which implies directly that

S approaches from below $\Rightarrow \omega_B < \omega_c$
T approaches from above $\Rightarrow \omega_{BT} > \omega_c$.

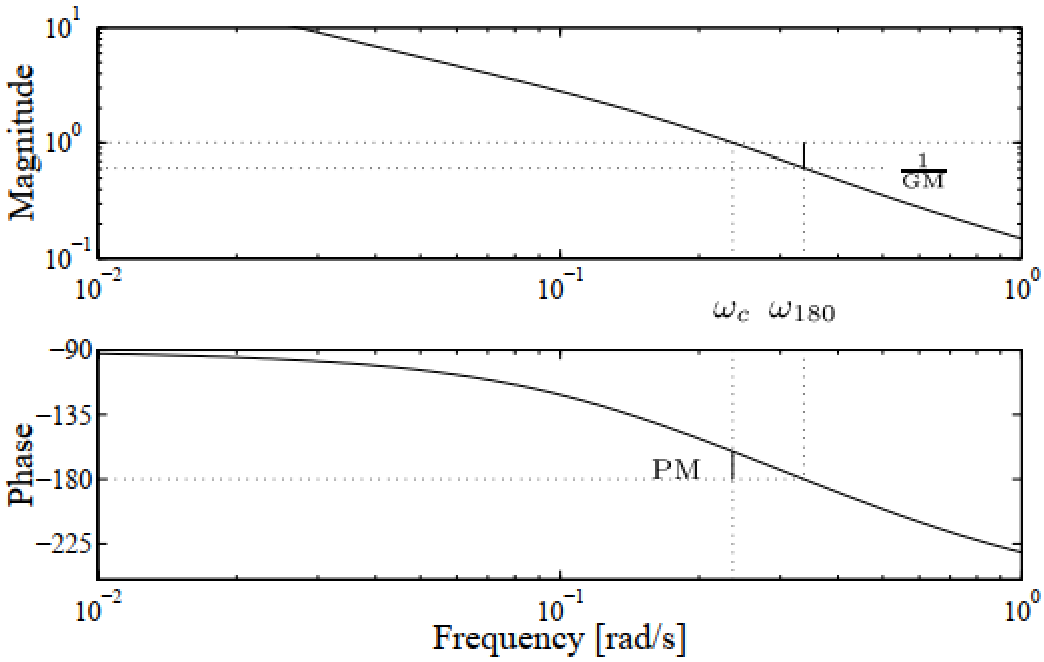
We can conclude that roughly all the frequencies described can be used to discuss bandwidth, describing the behaviour of the closed-loop system.

3. Consider a SISO-system. The maximum values of the sensitivity and complementary functions are denoted M_S and M_T, respectively. Let the gain and phase margins of a closed-loop system be *GM* (gain margin) and *PM* (phase margin). Prove that

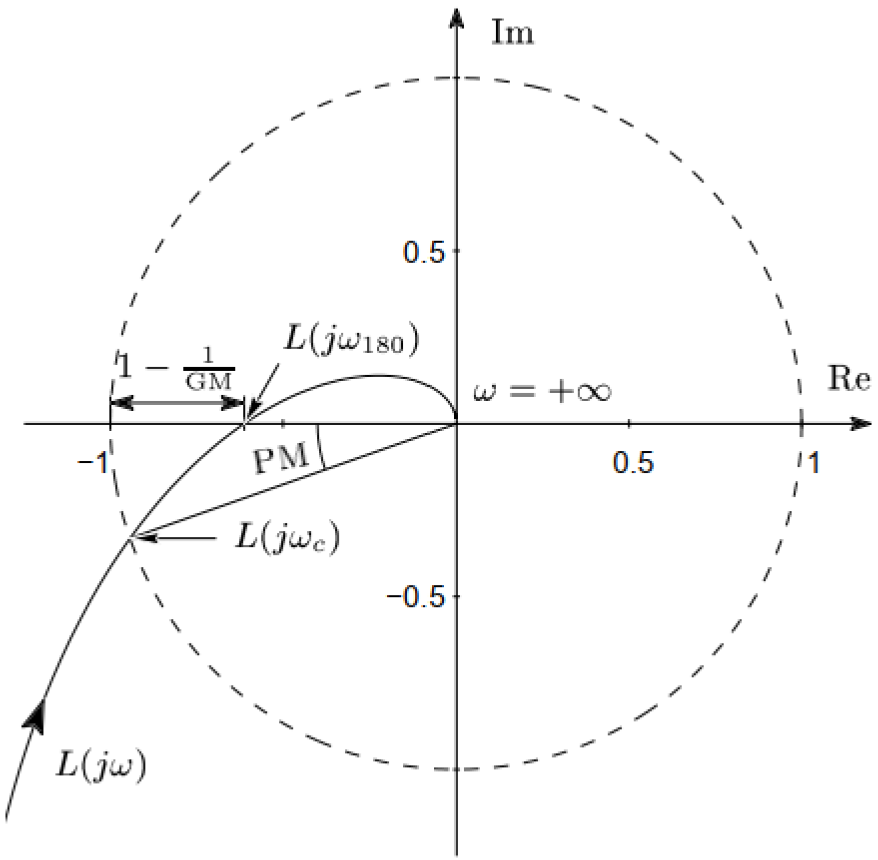
$$GM \geq \frac{M_S}{M_S - 1} \qquad PM \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} \text{ [rad]}$$

$$GM \geq 1 + \frac{1}{M_T} \qquad PM \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T} \text{ [rad]}$$

Solution:



Typical Bode plot of $L(j\omega)$ with PM and GM indicated.



Typical Nyquist plot of $L(j\omega)$ with PM and GM indicated. Closed-loop instability occurs, when $L(j\omega)$ encircles the critical point -1.

From the Bode plot we can see that

$$GM = \frac{1}{|L(j\omega_{180})|}$$

$$\angle L(j\omega_{180}) = -180^\circ$$

$$PM = \angle L(j\omega_c) + 180^\circ$$

$$|L(j\omega_c)| = 1$$

Denote the phase crossover frequency by ω_{180} (then the phase of L is –180 degrees). By the definition of the gain margin

$$GM = \frac{1}{|L(j\omega_{180})|} \quad \Rightarrow \quad L(j\omega_{180}) = \frac{-1}{GM}$$

We obtain

$$T(j\omega_{180}) = \frac{L(j\omega_{180})}{1+L(j\omega_{180})} = \frac{\frac{-1}{GM}}{-\frac{1}{GM}+1} = \frac{\frac{-1}{GM}}{-\frac{1}{GM}+\frac{GM}{GM}} = \frac{-1}{GM}\frac{GM}{GM-1} = \frac{-1}{GM-1}$$

$$S(j\omega_{180}) = \frac{1}{1+L(j\omega_{180})} = \frac{1}{1-\frac{1}{GM}}$$

Now use the abbreviations $M_T = \max_{\omega} |T(i\omega)|$, $M_S = \max_{\omega} |S(i\omega)|$

and it follows that

$$M_T \geq \frac{1}{|GM-1|}; \qquad M_S \geq \frac{1}{\left|1-\frac{1}{GM}\right|}$$

and the gain margin inequalities given in the problem follow easily. Let us calculate the first as an example.

$$M_T \geq \frac{1}{|GM-1|} \Rightarrow |GM-1| \geq \frac{1}{M_T} \Rightarrow GM-1 \geq \frac{1}{M_T} \Rightarrow GM \geq 1+\frac{1}{M_T}$$

The inequality related to M_S is derived correspondingly:

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$$M_S \geq \frac{1}{\left|1-\frac{1}{GM}\right|} \Leftrightarrow \left|1-\frac{1}{GM}\right| \geq \frac{1}{M_s} \Leftrightarrow 1-\frac{1}{GM} \geq \frac{1}{M_s}$$

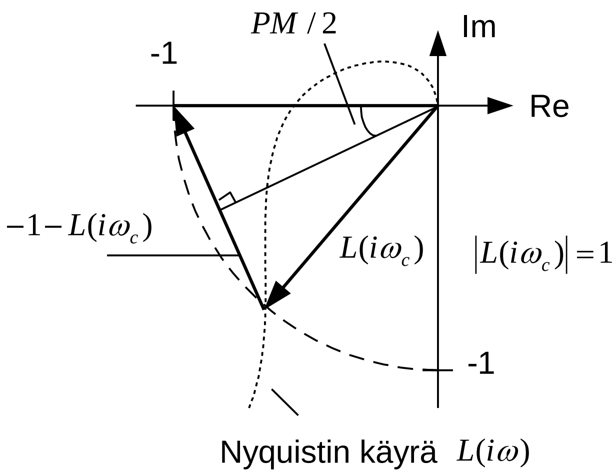
$$\Leftrightarrow 1-\frac{1}{M_s} \geq \frac{1}{GM} \Leftrightarrow \frac{M_s-1}{M_s} \geq \frac{1}{GM}$$

$$\Leftrightarrow GM \geq \frac{M_s}{M_s-1}$$

Considering the phase margin note that

$$|S(j\omega_c)| = \frac{1}{|1+L(j\omega_c)|} = \frac{1}{|-1-L(j\omega_c)|}$$

in which ω_c is the gain crossover frequency (the gain of L is one in this frequency).



From the figure it can be seen that

$$\sin(PM/2) = \frac{-\frac{1}{2}(-1-L(j\omega_c))}{1}$$

$$2\sin(PM/2) = 1 + L(j\omega_c)$$

$$|S(i\omega_c)| = |T(i\omega_c)| = \frac{1}{1+L(i\omega_c)} = \frac{1}{2\sin(PM/2)}$$

and the inequalities related to phase margin follow directly.

$$M_S = \max_{\omega} |S(i\omega)| = \frac{1}{|1+L(i\omega_c)|} \geq \frac{1}{2\sin(PM/2)}$$

$$\Leftrightarrow 2\sin(PM/2) \geq \frac{1}{M_s}$$

$$\Leftrightarrow PM/2 \geq \arcsin\left(\frac{1}{2M_s}\right)$$

$$\Leftrightarrow PM \geq 2\arcsin\left(\frac{1}{2M_s}\right)$$

(In the last form the following fact, obtained for example by the Taylor approximation, is used: when x is positive, $\arcsin(x) > x$.). This leads to the right hand side inequality

$$2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$$. Thus,

$$PM \geq 2\arcsin\left(\frac{1}{2M_s}\right) \geq \frac{1}{M_S}.$$

Similarly we can solve the inequality for $M_T = \max_{\omega} |T(j\omega)|$

The results show for example that if $M_T = 2$, then $GM \geq 1.5$, $PM \geq 29^\circ$.

Sometimes the maximum values (∞ - norms) M_S and M_T are used as alternatives to gain and phase margins. For example, demanding that $M_S < 2$, the often used ”rules of thumb” $GM > 2$, $PM > 30^\circ$ follow.