Lecture 11 The BCS theory of superconductivity continued, and the BCS-BEC crossover

Literature for the BCS theory: A.L. Fetter and J.D. Walecka, Quantum theory of many-particle systems, Dover, Chapters 36-37; P.G. De Gennes, Superconductivity of metals and alloys, Westview Press, Chapters 4-5
Literature for the BCS-BEC crossover: Chapter 9. "The BCS-BEC Crossover" by Meera M. Parish, and Chapter 10. "Spectroscopies - Theory" by Päivi Törmä in "Quantum Gas Experiments – Exploring Many-Body States", P. Törmä and K. Sengstock (Eds.), Ref. [1]. Meera Parish is thanked for providing the pictures for this lecture.

Learning goals

- To know about the BCS wave function and that it is possible to obtain the results also using variational ansatz.
- To know that the Fermi surface disappears in the BCS superconductor/superfluid, i.e. that it is not a Fermi liquid.
- To learn the connection between BCS-type superfluids of fermions and Bose-Einstein condensation of composite bosons.
- To be aware of some of the key quantum gas experiments that have proven the existence of the BCS-BEC crossover.

24.4 The BCS wave function

Apart from the diagonalization by the Bogoliubov transformation, an alternative route leading to the same end-results is a **variational calculation** where one makes a parametrized guess (ansatz) for the ground state of the system, calculates the expectation value of the Hamiltonian using the guess wave function, and then minimizes this expectation value, leading to certain values for the parameters. If the BCS ansatz gives a lower energy than the normal state wave function, then one knows that the normal state is not the ground state.

The BCS ansatz (the BCS wavefunction) is of the following form:

$$|\mathrm{BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow} \right) |0\rangle . \tag{24.48}$$

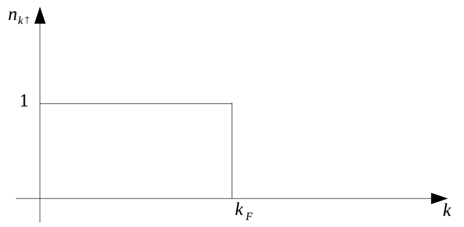
One can then calculate

$$\langle \mathrm{BCS}| H |\mathrm{BCS}\rangle \tag{24.49}$$

and minimize it (this is similar to the exercise related to the Gutzwiller ansatz and finding the superfluid and Mott insulator phases by minimizing the energy). The minimization gives values for the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, and they are the same as found above by the Bogoliubov transformation.

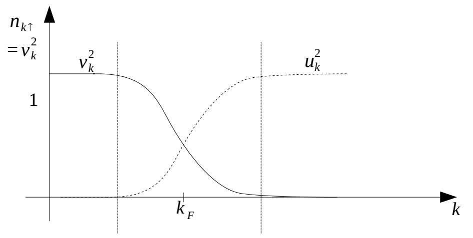
The BCS wave function is quite intuitive. For the noninteracting case, it becomes trivial: all the $v_{\mathbf{k}}$'s are unity below the Fermi level and zero above it ($v_{\mathbf{k}} = 1$ for $|\mathbf{k}| \leq |\mathbf{k}_F|$), and for $u_{\mathbf{k}}$ vice versa ($u_{\mathbf{k}} = 0$ for $|\mathbf{k}| \leq |\mathbf{k}_F|$). The momentum distributions show a sharp edge. A sharp feature at the edge tells about the existence of a well-defined **Fermi surface**:

$$|\mathrm{BCS}\rangle = \prod_{\mathbf{k} \leq \mathbf{k_F}} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow} |0\rangle . \tag{24.50}$$



$n_{k\uparrow} = \langle c^{+}_{k\uparrow} c_{k\uparrow} \rangle = v_k^2$ *(exercise)*

When there are interactions and a finite order parameter Δ, for certain momenta both the $u_{\mathbf{k}}$'s and $v_{\mathbf{k}}$'s are nonzero. The momentum distributions become smoothened and there is no more a well defined Fermi surface (sharp edge/drop in the momentum distribution like for the normal state or the Fermi liquid).



$\Delta \propto u_k v_k \rightarrow$ the particles around the Fermi level form Cooper pairs

25 Quantum phase transition vs. crossover

A crossover usually refers in physics to a situation where the system parameters change from one regime to another. For instance, the interactions in a system may change from weak to strong. Or, there can be a dimensional crossover from e.g. one to two-dimensional configuration. There can be also crossovers between two qualitatively different many-body states. In general, the word crossover refers to *smooth and continuous* evolution between the regimes or states. This is in contrast to quantum phase transitions (or usual thermal phase transitions). There the change from one system state to another is abrupt: thermodynamic quantities diverge. Remind yourself also about the difference between a usual thermal phase transition and a quantum phase transition: in the former, entropy and other system energies compete and the transition occurs at finite temperature (e.g. Ising model) while in the latter two different energies of the system (for instance kinetic and interaction energies) compete and the transition can happen also at zero temperature when a system parameter is changed (e.g. superfluid - Mott insulator transition).

Important

- (Quantum) phase transition: abrupt change in system state, thermodynamic quantities diverge

- Crossover: smooth evolution of quantities from one qualitatively different state/regime to another

26 The BCS-BEC crossover

In this course, you have learned about Bose-Einstein condensation (BEC) and about the BCS theory of superconductivity. The BCS theory presents superconductivity (superfluidity) as condensation of Cooper pairs. The theory describing the condensate of these pairs is, however, appears distinct from the theory of BEC of bosons: the many-body wavefunction and the excitation spectrum look quite different. However, one can argue that these two are connected. The Cooper pairs are actually just correlations in momentum space, so one can say that the radius of the pair is very large: so large that several Cooper pairs overlap spatially. Now think about tuning the interparticle interaction between the particles so that the pair size becomes smaller and smaller, eventually so small that a pair can be considered a point-like particle (that is, the pair size is much much smaller than the average interparticle distance in the system). That is, we have formed composite bosons. Now, are these bosons still condensed forming a BEC, and *is the evolution from Cooper pairs to BEC of composite bosons completely smooth or are there perhaps quantum phase transitions*? This is a question that the Nobel laureate Anthony Leggett and other researchers such as Eagles, Noziéres and Schmitt-Rink asked already in the 1980's and even earlier [2, 3, 4]. The actual answer to this question was obtained by ultracold gas systems only during the last decade. Let us now approach the question first from a simple mean-field many-body description point of view, following the original arguments of in Refs. [2, 3, 4], and a recent review by Meera Parish [5] (apart from the book, a (not fully up-to-date) version of the article is available in the arXiv, arXiv:1402.5171).

Let us consider a Fermi gas with two components ($\uparrow$, $\downarrow$) in three dimensions (3D), with the Hamiltonian that is the same as in the previous lecture:

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}\sigma} (\epsilon_{\mathbf{k}} - \mu)\, \hat{c}^{\dagger}_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}\sigma} + \frac{V_0}{V} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{q}-\mathbf{k}\downarrow}\hat{c}_{\mathbf{q}-\mathbf{k}'\downarrow}\hat{c}_{\mathbf{k}'\uparrow}, \tag{26.1}$$

where the spin $\sigma = \{\uparrow, \downarrow\}$, the momentum dispersion $\epsilon_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$, V is the system volume, and μ the chemical potential. An attractive contact interaction with the strength $V_0 < 0$ is considered. Note that here the mean-field approximation was not done, this is just the initial Hamiltonian (24.4), expressed with field operators, transformed to the momentum basis using the transformation of field operators to the momentum basis via a Fourier type transform, Eq.(24.16). Then performing one spatial integration in the Hamiltonian (24.4) imposes a restriction to the four momenta in the Fourier transforms and summation over three momenta are left. Now approaching a mean-field approximation from this transformed Hamiltonian, one notices that Hamiltonian implies also pairs with a finite momentum $\mathbf{q}$. In the BCS theory, it is assumed that a macroscopic number of pairs starts to accumulate with $\mathbf{q} = 0$ and the order parameter is chosen accordingly. This Hamiltonian can thus support both loosely bound Cooper pairs, as we have learned in previous lectures, and tightly bound molecule-type pairs which can be considered as composite bosons. We will come back to this point later, but for the moment, assume that these different types of pairs exist.

Let us now consider the composite boson regime. One can write the operator of a composite boson as

$$\hat{b}^{\dagger}_{\mathbf{q}} = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{\mathbf{q}-\mathbf{k}\downarrow} \tag{26.2}$$

where $\varphi_{\mathbf{k}}$ is the relative two-body wave function in momentum space. The many-body wave-function of a BEC of non-interacting bosons can be expressed as a coherent state:

$$|\Psi\rangle_{BEC} = \mathcal{N} e^{\lambda \hat{b}^{\dagger}_{0}} |0\rangle \tag{26.3}$$

where $\mathcal{N}$ is a normalization constant and $\lambda = \langle\Psi|\hat{b}_0|\Psi\rangle$ is the condensate order parameter. In other words, $|\lambda|^2/V$ corresponds to the condensate density. Note that this assumes a weakly interacting boson gas with negligible quantum depletion (that is, occupation of other than $\mathbf{k} = 0$ states even at $T = 0$) of the condensate due to interactions. This is a good approximation deep in the composite boson regime.

Using the definition of the composite boson operator (26.2) the state is

$$|\Psi\rangle_{BEC} = \mathcal{N} e^{\lambda \sum_{\mathbf{k}} \varphi_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}} |0\rangle. \tag{26.4}$$

On the other hand we have just learned that the BCS state can be expressed by the many-body wavefunction

$$|\Psi\rangle_{BCS} = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}) |0\rangle, \tag{26.5}$$

where $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$ due to normalization. The beautiful discovery of Eagles [2], Leggett [3] as well as Noziéres and Schmitt-Rink [4] was that the two simple mean-field many-body wavefunctions are actually the same, $|\Psi\rangle_{BEC} = |\Psi\rangle_{BCS} \equiv |\Psi\rangle$, that is,

$$|\Psi\rangle = \mathcal{N} e^{\lambda \sum_{\mathbf{k}} \varphi_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}} |0\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}) |0\rangle \tag{26.6}$$

if one requires $v_{\mathbf{k}}/u_{\mathbf{k}} = \lambda \varphi_{\mathbf{k}}$ and $\mathcal{N} = \prod_{\mathbf{k}} u_{\mathbf{k}}$. To prove the equality (26.6) you can either expand the exponential to a series, then remember that the particles are fermions so that higher powers of the operators become zero, and reorganize the terms to be of the form of the right hand side of the equation. Or, you can first prove by commutation relations that it is justified to write the sum in the exponential as a product, and then proceed similarly. You will do this in **Exercise Set 12**. This means that the BEC regime where composite bosons condense and the BCS regime of condensation of Cooper pairs are described by one wavefunction that evolves smoothly from one regime to the other. That is, there is a **BCS-BEC crossover** — instead of, for instance, a quantum phase transition between the regimes. Of course, this was just a prediction based on approximate mean-field theory which may be inadequate especially in the regime between the BCS and BEC sides. Therefore, it was a fundamentally important achievement to show by quantum gas experiments that it is indeed a crossover. The simple theory given above actually describes the crossover qualitatively quite well at low temperatures. However, it is not sufficient for quantitative predictions, especially at elevated temperatures; other methods such as quantum Monte Carlo (QMC) calculations have to be applied and there are still many open questions related to the theory of the crossover regime at high temperatures.

26.1 Connection to scattering length and the Feshbach resonance

Now, let us relate the above discussion more closely to ultracold gas experiments. Interactions between atoms in ultracold gas systems are often characterized by the s-wave scattering length a_S. The s-wave is for most atoms the only relevant scattering channel at the ultralow temperatures. Furthermore, since the gases are dilute, the range of the potential is usually very small compared to the interparticle distance and therefore the (in principle distance-dependent and complicated) potential can be approximated by a contact potential characterized by a single quantity, the scattering length.

The bare interaction V_0 in the Hamiltonian (26.1) is related to the scattering length a_S via the renormalization relation we discussed in last lecture (renormalization is needed due to the unphysical assumption of a contact interaction)

$$\frac{m}{4\pi\hbar^2 a_S} = \frac{1}{V_0} + \frac{1}{V}\sum_{\mathbf{k}}^{\Lambda} \frac{1}{2\epsilon_{\mathbf{k}}}. \tag{26.7}$$

Here we have introduced an ultraviolet cut-off Λ which will be the upper limit of the summation (integral). Its physical meaning can be understood as the inverse of the range of the interaction potential: in effect, we now put back in the lengthscale that was neglected when a contact interaction with zero range was done. The diluteness and low energies of the ultracold collisions mean that $\Lambda \gg 1/a_S$, $\Lambda \gg k_F$. Formally, one can take the limits $V_0 \to 0$, $\Lambda \to \infty$ while keeping the left hand side of Eq. (26.7) fixed and finite. The value of the bare interaction is thus irrelevant as well as the cutoff, and by scaling the Hamiltonian by the Fermi energy (a nice exercise) one sees that the ground state of the system depends on a single dimensionless parameter $k_F a_S$. The phenomenon of Feshbach resonance can be used to tune a_S. A two-body bound state appears around $1/a_S = 0$. The BCS and BEC regimes are reached, respectively, in the limits $-1/k_F a_S \gg 1$ and $1/k_F a_S \gg 1$. The crossover region, or so called unitarity region, is defined as $k_F|a_S| > 1$. Thus by tuning the scattering length by magnetic field, one can experimentally explore the BCS-BEC crossover.

26.2 Momentum distribution

Let us now go back to the BCS-BEC wavefunction (26.6). One can do a mean-field description of the problem, corresponding to this wavefunction (c.f. earlier lectures) and use the scattering length a_s in the Hamiltonian instead of the bare interaction. The momentum distribution $n_k = |v_k|^2$ then evolves throughout the crossover as depicted in Figure 16. In the BCS regime, the Fermi surface is only slightly smoothened due to pairing for momenta around the Fermi momentum. Towards the BEC regime, the distributions become flatter and flatter; more and more momentum states are needed to build up tightly bound pairs (in the Fourier sense, when the two fermions localize very close to each other, a wide range of momenta of the individual particles is needed to create such a localized entity).

26.3 Ground state properties

In the ground state, the free energy $\Omega = \langle\Psi|\hat{H} - \mu\hat{N}\,|\Psi\,\rangle$ is minimized. The free energy is

$$\Omega = 2\sum_{\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu)|v_{\mathbf{k}}|^2 + \frac{V_0}{V}\sum_{\mathbf{k}\mathbf{k}'} v_{\mathbf{k}}^* u_{\mathbf{k}} v_{\mathbf{k}'} u_{\mathbf{k}'}^* + \frac{V_0}{V}\sum_{\mathbf{k}\mathbf{k}'} |v_{\mathbf{k}}|^2 |v_{\mathbf{k}'}|^2. \tag{26.8}$$

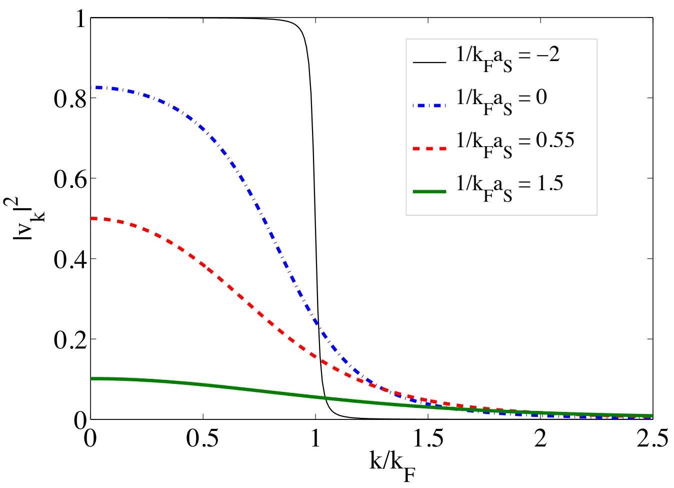


Figure 16: The momentum distribution $v_{\mathbf{k}}^2$ for interactions $1/k_F a_S$ varied across the BCS–BEC crossover.

We take u, v to be real as before. The last term in the free energy corresponds to the mean-field Hartree energy $V_0 n^2$ which can be neglected since for short range interactions the bare interaction V_0 can be considered small (remember in the previous lecture we combined the Hartree term with the chemical potential which could obviously be done also here). Minimizing the free energy at fixed chemical potential (take a derivative with respect to $v_{\mathbf{k}}$ and use $u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$ to get $du_{\mathbf{k}}/(dv_{\mathbf{k}})$) gives

$$2(\epsilon_{\mathbf{k}} - \mu)u_{\mathbf{k}}v_{\mathbf{k}} + (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\frac{V_0}{V}\sum_{\mathbf{k}'} u_{\mathbf{k}'}v_{\mathbf{k}'} = 0. \tag{26.9}$$

Remarkably, in the limit $v_{\mathbf{k}} \to 0$, where the effects of Pauli exclusion should be negligible, this reduces to the Schrödinger equation (just replace $u_{\mathbf{k}}$ and $u_{\mathbf{k}}^2$ by one) for the two-body bound state with wave function $v_{\mathbf{k}}/\sqrt{N}$ and binding energy -2μ:

$$2(\epsilon_{\mathbf{k}} - \mu)v_{\mathbf{k}} + \frac{V_0}{V}\sum_{\mathbf{k}'} v_{\mathbf{k}'} = 0. \tag{26.10}$$

This is another way of seeing how the wave function familiar from the BCS context also produces (approximately) something that describes a bound boson (or a condensate of them). In the BEC regime thus the chemical potential becomes directly related to the binding energy of the bound pair, $\mu \to -\varepsilon_B/2$.

When not taking the limit $v_{\mathbf{k}} \to 0$, one has to solve the following equations originating from Eq.(26.9):

$$\Delta \equiv \frac{V_0}{V}\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}} = -\frac{V_0}{V}\sum_{\mathbf{k}} \frac{\Delta}{2E_{\mathbf{k}}} \tag{26.11}$$

$$n = \frac{1}{V}\sum_{\mathbf{k}} v_{\mathbf{k}}^2 = \frac{1}{2V}\sum_{\mathbf{k}}\left(1 - \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right) \tag{26.12}$$

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2} \tag{26.13}$$

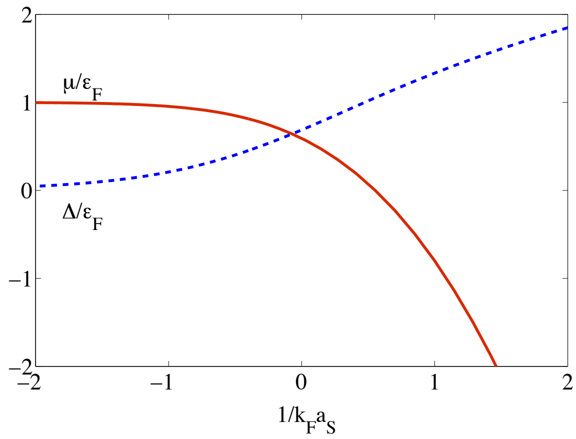


Figure 17: The chemical potential μ/ε_F and the order parameter Δ/ε_F througout the crossover.

Here Eqs. (26.11) and (26.12) correspond, respectively, to the usual gap and number equations which we have seen in the previous lecture, and Δ is the BCS order parameter. Note that sometimes in context of BCS theory especially for low temperature superconductors, only the gap equation is solved. However, for large interactions, this starts to produce unphysically large order parameters (gap energies) Δ. To keep these in control when approching strong interactions, it is necessary to solve also the number equation self-consistently together with the gap equation. This is sometimes called the BCS-Leggett theory, and it can describe the BCS-BEC crossover qualitatively well near $T = 0$.

It is of interest to consider the single particle excitation energy $E_{\mathbf{k}}$ in the two regimes of the crossover (note that we could have obtained $E_{\mathbf{k}}$ also by doing a Bogoliubov transformation to the mean-field Hamiltonian, as done in the previous lecture). In the BCS regime, chemical potential is positive and the excitation energy becomes essentially the BCS gap energy Δ for momenta such that $\epsilon_{\mathbf{k}} \sim \mu$ i.e. close to the Fermi level. This is the single particle energy gap known from the BCS theory. On the BEC side the chemical potential becomes negative and large in absolute value, since it corresponds to the binding energy. The order parameter Δ still remains finite, but the chemical potential dominates the single particle excitation energy $E_{\mathbf{k}}$ so that it becomes essentially $\mu_B/2$ i.e. half of the binding energy. The single particle excitations in the BEC regime thus involve breaking a tightly bound pair. The development of the order parameter and the chemical potential throughout the crossover are depicted in Figure 17.

We have thus seen that both the momentum distribution (Figure 16) and the chemical potential and order parameter (Figure 17) evolve smoothly from the BCS to the BEC regimes. It is also easy to see from the analytical form of the order parameter that indeed the order parameters at the two ends are connected. In the BCS regime, the order parameter is

$$\Delta \propto \sum_{\mathbf{k}} \langle \hat{c}^\dagger_{\mathbf{k}\uparrow} \hat{c}^\dagger_{-\mathbf{k}\downarrow} \rangle. \tag{26.14}$$

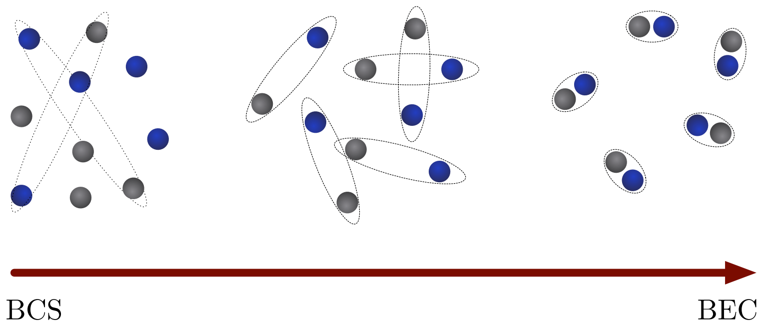


Figure 18: Change in the pair size in the BCS-BEC crossover in a two-component Fermi gas.

On the other hand, in the BEC regime the order parameter is

$$\langle\Psi|\hat{b}_0\,|\Psi\rangle = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} \langle \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}\rangle. \tag{26.15}$$

For $\varphi_{\mathbf{k}}$ that is nearly the same for all $\mathbf{k}$, which is the case for the deep BEC limit where the momentum distribution becomes flat, this indeed approaches

$$\langle\Psi|\hat{b}_0\,|\Psi\rangle \propto \sum_{\mathbf{k}} \langle \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}\rangle \tag{26.16}$$

which is of the same form as the BCS order parameter.

Apart from single particle excitations, there is a low energy bosonic collective mode (gapless Goldstone mode) associated with fluctuations of Δ. This mode could be calculated by introducing interactions between the quasiparticles. This collective mode also evolves smoothly from one regime to another. Since the binding energy and thus the single particle excitation energy is very large in the BEC regime, the collective mode is then the only low-energy excitation.

26.4 Unitarity regime

We have now seen how the two-component Fermi gas can evolve from Cooper pairs, which are correlated mainly in the momentum space and thus overlap physically, to tightly bound pairs in the BEC regime. Figure 18 shows this schematically; in the crossover regime $|k_F a_S| > 1$, the pair size becomes of the order of the inter-particle spacing. One may anticipate that this can lead to some interesting new physics. The limit $1/k_F a_S = 0$ is called the **unitarity limit** and it gives rise to a **universal** strongly interacting Fermi gas which is independent of any interaction length scale. That is, formally the scattering length goes to infinity, but that actually does not make any of the physical quantities singular. Instead of the scattering length, the system is characterized by a universal constant ξ. At zero temperature, all thermodynamic quantities only depend on density via ξ; for instance, the chemical potential $\mu = \xi\varepsilon_F$ and the total energy $E = \xi\frac{3}{5}\varepsilon_F N$. Ultracold gases were the first realization of the concept of a unitary Fermi gas. For those who wish to study the topic in depth, Ref. [6] is recommended.

In addition to $1/k_F a_S = 0$, there is another special point in the unitarity regime, namely the point where $\mu = 0$. This is a turning point since the quasiparticle energy $E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}$ has a minimum at a finite k for positive μ and at $k = 0$ for negative chemical potential, the former corresponding to the BCS side and the latter to the BEC of bound pairs. In fact, $\mu = 0$ can be understood as the point where the Fermi surface disappears; the boundary between the BCS and BEC regimes. It does not occur exactly at the Feshbach resonance point $1/k_F a_S = 0$ but a bit on the repulsive interaction (positive scattering length) side of the resonance, namely $1/k_F a_S \simeq 0.55$ as is seen from Figure 17. For those who wish to understand this topic deeper (not part of the course), I recommend the article [7] where a simple calculation on two-particle scattering restricted by a Fermi sea also shows that the bound state appears for a finite value of the positive scattering length.

26.5 Finite temperature

The transition temperature is determined by the low-energy excitations of the condensate. In the BCS regime, the pairing gap is small and the single particle excitations that break Cooper pairs are the relevant low-energy excitations. The critical temperature of pair condensation thus coincides with the one of pair formation. By minimizing the mean-field free energy at finite temperature, and by seeing at which temperature the gap Δ disappears

$$\Omega(T) = -V\frac{\Delta^2}{V_0} + \sum_{\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu - E_{\mathbf{k}}) - 2k_B T \sum_{\mathbf{k}} \ln\left(1 + e^{-E_{\mathbf{k}}/k_B T}\right) \qquad (26.17)$$

one can show that $T_c \sim \Delta(0)$ (here $k_B = 1$) and thus T_c/ε_F goes to zero exponentially when $1/k_F a_S \to -\infty$ as known from BCS theory.

In the BEC regime, the energy needed for pair breaking and thus single particle excitations is very large, while the collective modes provide the low-energy excitations that determine the transition temperature. Thus the critical temperature for pairing T^* given by mean-field theory no longer coincides with T_c. In the limit $1/k_F a_S \to \infty$, T_c/ε_F saturates to the transition temperature for a non-interacting BEC, where $T_c/\varepsilon_F \simeq 0.218$. Describing the gas at finite temperature and in the strongly interacting unitarity regime is a major open theoretical challenge. Especially the nature of the normal state just above the critical temperature is not known. It might be, for instance, a Fermi liquid, or a so called pseudo-gap state which has gap-formation without superfluidity even above T_c. Ultracold gas experiments have not yet been able to unanimously decide between these options. The question is of fundamental imporatance due to the universal nature of the gas, as well as because some high-temperature superfluids are believed to be in the strong interaction regime between the BCS and BEC ends. Indeed the transition temperature seems to be high at the unitarity regime, see Figure 19.

26.6 Experiments on the BCS-BEC crossover

It is impossible to review exhaustively the ultracold gas experiments related to the BCS-BEC crossover in this lecture; I refer to Refs. [1, 5, 6] for more references. Note that the problem has been approached also in other systems than ultracold gases although they offer so far the only clean and controllable platform to study the crossover. Here I mention only some examples.

The first important crossover studies appeared in the year 2004 when two-component strongly interacting Fermi gases were cooled down and manipulated with the Feshbach resonance in such a way that condensates of molecules formed of the two components (bound pairs) were achieved. This was done simultaneously

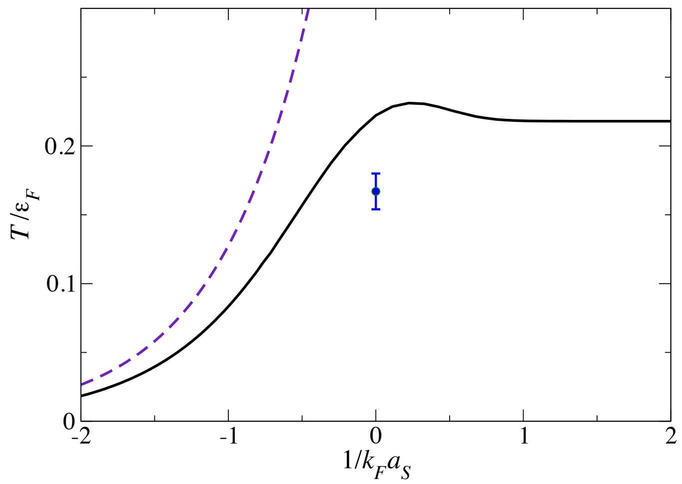


Figure 19: Condensation transition temperature T_c throughout the BCS–BEC crossover, calculated using the Nozières–Schmitt-Rink approach [4] which includes Gaussian fluctuations. The dashed line marks the temperature T^* around which pairs start to form. The filled circle marks the experimentally measured T_c at unitarity [8], which is consistent with quantum Monte Carlo predictions [9, 10]. Figure from [5].

by the groups of Rudi Grimm (Innsbruck) and Debbie Jin (JILA) [11, 12] and very soon after by the group of Nobel laureate Wolfgang Ketterle [13]. These experiments showed that a condensate is possible on the BEC side of the Feshbach resonance. In 2004, the groups of Jin [12] and Ketterle [14] showed the existence of Fermion pairs at the unitarity regime between the BCS and BEC regimes, and the experiments of Grimm's group using radio-frequency spectroscopy revealed the many-body nature of the pairing at the unitarity regime by showing that the pairing energy depended on the Fermi energy [15, 16]. Studies of the whole crossover then followed from the groups of John Thomas [17], Rudi Grimm [18], Christophe Salomon [19] and Randy Hulet [20] focusing e.g. on characterizing collective modes and pairing. The proof that the system is indeed a superfluid came in 2005 from Wolfgang Ketterle's group with the observation of quantized vortices in the gas [21]. All the experiments showed that the evolution from the BCS to BEC side is smooth and continuous, and thus indeed it is a crossover. The was a remarkable achievement proving correct the predictions of Eagles and Leggett.

The early research opened an avenue for a a large number of beautiful experiments studying the crossover physics in depth – it would take too much time to describe all of them in these lectures (one by M. Zwierlein's group where for instance T_c at unitarity was measured [8] was mentioned already in the context of Figure 19).

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