Problem 8.1: Robust LP with Polyhedral Uncertainty

Consider the following *robust* linear optimization problem with *polyhedral uncertainty*:

$$\min_{x} \; c^\top x \tag{1}$$

$$\text{subject to: } \max_{a_i \in \mathcal{P}_i} \; a_i^\top x \le b_i, \quad i = 1, \dots, m \tag{2}$$

with decision variables $x \in \mathbb{R}^n$ and polyhedral sets

$$\mathcal{P}_i = \{a_i : C_i a_i \le d_i\}, \text{ for all } i = 1, \dots, m.$$

The problem data are $c \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{m_i \times n}, a_i \in \mathbb{R}^n, d_i \in \mathbb{R}^{m_i}$, and $b \in \mathbb{R}^m$. We assume that the polyhedral sets $\mathcal{P}_i$ are nonempty for all $i = 1, \dots, m$. Notice that the problem (1) – (2) is an example of a bilevel optimization problem that we studied in Exercise 7.

Show that the problem (1) – (2) is equivalent to the following linear optimization problem:

$$\min_{x,u} \; c^\top x \tag{3}$$

$$\text{subject to: } d_i^\top u_i \le b_i, \; i = 1, \dots, m \tag{4}$$

$$C_i^\top u_i = x, \; i = 1, \dots, m \tag{5}$$

$$u_i \ge 0, \; i = 1, \dots, m \tag{6}$$

with variables $x \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^{m_i}$ for all $i = 1, \dots, m$.

Hint: Replace the inner optimization problems in the constraints (2):

$$\max_{a_i \in \mathcal{P}_i} \; a_i^\top x, \; i = 1, \dots, m \tag{7}$$

by writing their Lagrangian dual problems with dual variables u_i for all $i = 1, \dots, m$.

Problem 8.2: Lagrangian of a Quadratic Optimization Problem

Consider the following quadratic optimization problem with inequality constraints:

$$\min_{x} \; \frac{1}{2} x^\top H x + d^\top x \tag{8}$$

$$\text{subject to: } Ax \le b \tag{9}$$

with decision variables $x \in \mathbb{R}^n$. The problem data are $d \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $H \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. The objective function

$$f(x) = \frac{1}{2} x^\top H x + d^\top x$$

is thus a strictly convex function (why?). Write the Lagrangian dual problem of of (8) – (9) and derive its dual explicitly.

Problem 8.3: Duality in Linear Optimization

Consider the following linear optimization problem:

$$\min_{x} \; c^\top x \tag{10}$$

$$\text{subject to: } Ax = b \tag{11}$$

$$x \geq 0 \tag{12}$$

with decision variables $x \in \mathbb{R}^n$. The problem data are $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. We will call (10) – (12) the *primal* problem.

(a) Derive the dual problem of the primal (10) – (12) by using Lagrangian duality.

(b) Show that the dual of the dual problem derived in part (a) is equivalent to the primal problem (10) – (12). *Hint:* Use Lagrangian duality.

(c) Show that weak duality holds between the primal (10) – (12) and its dual problem.