Problem 8.1: Robust LP with Polyhedral Uncertainty

Consider the following *robust* linear optimization problem with *polyhedral uncertainty*:

$$\min_{x} \; c^\top x \tag{1}$$

$$\text{subject to: } \max_{a_i \in \mathcal{P}_i} a_i^\top x \leq b_i, \quad i = 1, \dots, m \tag{2}$$

with decision variables $x \in \mathbb{R}^n$ and polyhedral sets

$$\mathcal{P}_i = \{a_i : C_i a_i \leq d_i\}, \text{ for all } i = 1, \dots, m.$$

The problem data are $c \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{m_i \times n}, a_i \in \mathbb{R}^n, d_i \in \mathbb{R}^{m_i}$, and $b \in \mathbb{R}^m$. We assume that the polyhedral sets $\mathcal{P}_i$ are nonempty for all $i = 1, \dots, m$. Notice that the problem (1) – (2) is an example of a bilevel optimization problem that we studied in Exercise 7.

Show that the problem (1) – (2) is equivalent to the following linear optimization problem:

$$\min_{x,u} \; c^\top x \tag{3}$$

$$\text{subject to: } d_i^\top u_i \leq b_i, \; i = 1, \dots, m \tag{4}$$

$$C_i^\top u_i = x, \; i = 1, \dots, m \tag{5}$$

$$u_i \geq 0, \; i = 1, \dots, m \tag{6}$$

with variables $x \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^{m_i}$ for all $i = 1, \dots, m$.

Hint: Replace the inner optimization problems in the constraints (2):

$$\max_{a_i \in \mathcal{P}_i} \; a_i^\top x, \; i = 1, \dots, m \tag{7}$$

by writing their Lagrangian dual problems with dual variables u_i for all $i = 1, \dots, m$.

Solution.

We can express the problem (1) – (2) as

$$\min_{x} \; c^\top x \tag{8}$$

$$\text{subject to: } g_i(x) \leq b_i, \quad i = 1, \dots, m, \tag{9}$$

where $g_i(x)$ is the optimal value of the linear optimization problem

$$g_i(x) = \max_{a_i} \; a_i^\top x \tag{10}$$

$$\text{subject to: } C_i a_i \leq d_i, \tag{11}$$

for all $i = 1, \dots, m$. The Lagrangian dual function of (10) – (11) is

$$\theta(u_i) = \max_{a_i} \; \{a_i^\top x + u_i^\top (d_i - C_i a_i)\} \tag{12}$$

and the corresponding Lagrangian dual problem is

$$\min_{u_i} \; \theta(u_i) \tag{13}$$

$$\text{subject to: } u_i \geq 0. \tag{14}$$

We can write the problem (13) – (14) as

$$\min_{u_i} \; \{\max_{a_i} \; \{x^\top a_i + u_i^\top (d_i - C_i a_i)\}\}$$

$$\text{subject to: } u_i \geq 0,$$

which can be rewritten as

$$\begin{aligned}\min_{u_i}.\ &\{\max_{a_i}.\ \{x^\top a_i + u_i^\top d_i - u_i^\top C_i a_i\}\}\\ \text{subject to: }&u_i \geq 0\end{aligned}$$

or

$$\min_{u_i}.\ \{d_i^\top u_i + \{\max_{a_i}.\ (x^\top - u_i^\top C_i)a_i\}\} \quad (15)$$
$$\text{subject to: } u_i \geq 0. \quad (16)$$

Now, since the values of a_i are not restricted, the value of the inner maximization problem in (15) becomes ∞ unless $x^\top - u_i^\top C_i = 0$ or $C_i^\top u_i = x$. Thus, we can write (15) – (16) as the following linear optimization problem

$$\min_{u_i}.\ d_i^\top u_i \quad (17)$$
$$\text{subject to: } C_i^\top u_i = x \quad (18)$$
$$u_i \geq 0, \quad (19)$$

which is the dual of (10) – (11). As strong duality holds between linear optimization problems, the optimal value of (17) – (19) is equal to $g_i(x)$ in (10) – (11). Therefore, we have $g_i(x) \leq b_i$ in (9) if and only if there exists a u_i with

$$d_i^\top u_i \leq b_i, \qquad C_i^\top u_i = x, \qquad u_i \geq 0 \quad (20)$$

for all $i = 1, \dots, m$. Now, replacing $g_i(x)$ in (8) – (9) with the constraints (20) for all $i = 1, \dots, m$, we finally get

$$\begin{aligned}\min_{x,u}.\ &c^\top x\\ \text{subject to: }&d_i^\top u_i \leq b_i,\ i = 1, \dots, m\\ &C_i^\top u_i = x,\ i = 1, \dots, m\\ &u_i \geq 0,\ i = 1, \dots, m.\end{aligned}$$

Problem 8.2: Lagrangian of a Quadratic Optimization Problem

Consider the following quadratic optimization problem with inequality constraints:

$$\min_{x}.\ \frac{1}{2}x^\top Hx + d^\top x \quad (21)$$
$$\text{subject to: } Ax \leq b \quad (22)$$

with decision variables $x \in \mathbb{R}^n$. The problem data are $d \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $H \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. The objective function

$$f(x) = \frac{1}{2}x^\top Hx + d^\top x$$

is thus a strictly convex function (why?). Write the Lagrangian dual problem of of (21) – (22) and derive its dual explicitly.

Solution

The objective function is strictly convex since its Hessian $\nabla^2 f(x) = H > 0$ for all $x \in \mathbb{R}^n$, which is a necessary and sufficient condition for strict convexity. The Lagrangian dual function of (21) – (22) can be written as

$$\theta(u) = \min_{x}.\ \left\{\frac{1}{2}x^\top Hx + d^\top x + u^\top(Ax - b)\right\}$$

and the corresponding Lagrangian dual problem becomes

$$\max_u. \ \{\theta(u) : u \geq 0\}$$

or

$$\max_u. \ \left\{-u^\top b + \min_x. \ \left\{\frac{1}{2}x^\top Hx + d^\top x + u^\top Ax\right\}\right\} \tag{23}$$

$$\text{subject to: } u \geq 0 \tag{24}$$

Let us define

$$g(x) = \frac{1}{2}x^\top Hx + d^\top x + u^\top Ax$$

which allows us to write the inner minimization problem of (23) as

$$\min_x. \quad g(x) \tag{25}$$

We can solve the inner minimization problem (25) by taking the gradient of $g(x)$ and setting it to zero:

$$\begin{aligned}\nabla g(x) &= \nabla(\frac{1}{2}x^\top Hx + d^\top x + u^\top Ax) \\ &= Hx + d + A^\top u = 0.\end{aligned} \tag{26}$$

Since $H > 0$ is invertible, we get the unique optimal primal solution from (26):

$$x = -H^{-1}(d + A^\top u) \tag{27}$$

Moreover, by multiplying (26) with $x^\top$ we get

$$x^\top Hx + x^\top d + x^\top A^\top u = 0$$

which can be written as

$$x^\top Hx + d^\top x + u^\top Ax = 0 \tag{28}$$

Now, by first substituting (28) to the Lagrangian dual problem (23) – (24), we get

$$\max_u. \ \left\{-u^\top b + \min_x. \ \left\{-\frac{1}{2}x^\top Hx\right\}\right\} \tag{29}$$

$$\text{subject to: } u \geq 0 \tag{30}$$

and by further substituting (27) to (29) – (30), the Lagrangian dual becomes

$$\max_u. \ \left\{-u^\top b - \frac{1}{2}(-H^{-1}(d + A^\top u))^\top H(-H^{-1})(d + A^\top u)\right\} \tag{31}$$

$$\text{subject to: } u \geq 0 \tag{32}$$

Let us define the function inside (31) as

$$h(u) = -u^\top b - \frac{1}{2}(-H^{-1}(d + A^\top u))^\top H(-H^{-1})(d + A^\top u)$$

Noticing that $(H^{-1})^\top = H^{-1}$ since H is symmetric, we can further simplify $h(u)$ as:

$$\begin{aligned}
h(u) &= -u^\top b - \frac{1}{2}(d + A^\top u)^\top H^{-1}(d + A^\top u) \\
&= -u^\top b - \frac{1}{2}(d^\top + u^\top A)(H^{-1}d + H^{-1}A^\top u) \\
&= -u^\top b - \frac{1}{2}(d^\top H^{-1}d + d^\top H^{-1}A^\top u + u^\top AH^{-1}d + u^\top AH^{-1}A^\top u) \\
&= -u^\top b - \frac{1}{2}(d^\top H^{-1}d + u^\top AH^{-1}d + u^\top AH^{-1}d + u^\top AH^{-1}A^\top u) \\
&= -u^\top b - \frac{1}{2}(d^\top H^{-1}d + 2u^\top AH^{-1}d + u^\top AH^{-1}A^\top u) \\
&= -u^\top b - u^\top AH^{-1}d - \frac{1}{2}d^\top H^{-1}d - \frac{1}{2}u^\top AH^{-1}A^\top u \\
&= -u^\top(b + AH^{-1}d) - \frac{1}{2}u^\top AH^{-1}A^\top u - \frac{1}{2}d^\top H^{-1}d \qquad (33)
\end{aligned}$$

Now, by defining

$$D = AH^{-1}A^\top \quad \text{and} \quad c = b + AH^{-1}d$$

we can write (33) as

$$h(u) = -c^\top u - \frac{1}{2}u^\top Du - \frac{1}{2}d^\top H^{-1}d \qquad (34)$$

and substituting (34) back to the Lagrangian dual problem (31), it becomes

$$\max_u. \quad \left\{-c^\top u - \frac{1}{2}u^\top Du - \frac{1}{2}d^\top H^{-1}d : u \geq 0\right\} \qquad (35)$$

Now, we can further notice that $(-1/2)d^\top H^{-1}d$ is a constant, so it has no effect on the optimization problem (35). We can thus finally write the dual (35) as

$$\begin{aligned}
\max_u. \; & -c^\top u - \frac{1}{2}u^\top Du \\
\text{subject to: } & u \geq 0
\end{aligned}$$

or, as a minimization problem:

$$\begin{aligned}
\min_u. \; & \frac{1}{2}u^\top Du + c^\top u \\
\text{subject to: } & u \geq 0
\end{aligned}$$

which is of similar form as the primal problem but simpler constraint structure.

Problem 8.3: Duality in Linear Optimization

Consider the following linear optimization problem:

$$\begin{aligned}
\min_x. \; & c^\top x && (36) \\
\text{subject to: } & Ax = b && (37) \\
& x \geq 0 && (38)
\end{aligned}$$

with decision variables $x \in \mathbb{R}^n$. The problem data are $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. We will call (36) – (38) the *primal* problem.

(a) Derive the dual problem of the primal (36) – (38) by using Lagrangian duality.

(b) Show that the dual of the dual problem derived in part (a) is equivalent to the primal problem (36) – (38). *Hint:* Use Lagrangian duality.

(c) Show that weak duality holds between the primal (36) – (38) and its dual problem.

Solution.

(a) The Lagrangian dual function of (36) – (38) can be written as

$$\theta(v) = \underset{x \geq 0}{\text{min.}} \ \left\{ c^\top x - v^\top (Ax - b) \right\} \tag{39}$$

and the corresponding Lagrangian dual problem becomes

$$\underset{v}{\text{max.}} \ \theta(v) = \underset{v}{\text{max.}} \ \left\{ v^\top b + \underset{x \geq 0}{\text{min.}} \ \left\{ (c - A^\top v)^\top x \right\} \right\} \tag{40}$$

In (40), the inner term must be non-negative: $c - A^\top v \geq 0$, because otherwise the value of the inner minimization problem becomes $-\infty$. Thus, the dual of the LP problem (36) – (38) becomes

$$\underset{v}{\text{max.}} \ v^\top b \tag{41}$$

$$\text{subject to: } A^\top v \leq c \tag{42}$$

(b) The Lagrangian dual function of the dual (41) – (42) can be written as

$$\theta(x) = \underset{v}{\text{max.}} \ \left\{ v^\top b + x^\top (c - A^\top v) \right\} \tag{43}$$

and the corresponding dual problem becomes

$$\underset{x \geq 0}{\text{min.}} \ \theta(x) = \underset{x \geq 0}{\text{min.}} \ \left\{ c^\top x + \underset{v}{\text{max.}} \ \left\{ (b - Ax)^\top v \right\} \right\} \tag{44}$$

Since the variable vector v is unrestricted, the inner maximization problem becomes ∞ unless $b - Ax = 0$. Thus, the dual of the dual problem becomes

$$\begin{aligned} &\underset{x}{\text{min.}} \ c^\top x \\ &\text{subject to: } Ax = b \\ &\qquad\qquad\quad x \geq 0 \end{aligned}$$

which is exactly the original primal problem.

(c) For any pair of feasible primal and dual solutions x and v, respectively, we have

$$v^\top b = v^\top Ax = x^\top A^\top v \leq x^\top c = c^\top x$$

which is exactly the definition of weak duality.