Exercise sheet 5

1. Let $f = u + iv$ be analytic in a domain D. Assume that $v(z) = u(z)^2$ for all $z \in D$. Prove that f is a constant function. (Hint: Differentiate $u(z)^2 - v(z)$ with respect to x and y.)

Solution

We have $u(z)^2 - v(z) = 0$ for all $z = x + iy$ in D. Differentiate first with respect to x.

We get (i) $2uu_x - v_x = 0$

Now with respect to y

(ii) $2uu_y - v_y = 0$

Using the CR-equations $u_x = v_y$; $u_y = -v_x$ we get

(i) $2uu_x + u_y = 0$

(ii) $2uu_y - u_x = 0$

Multiply (i) by u_y and (ii) by $-u_x$ and get

(i) $2uu_xu_y + u_y^2 = 0$

(ii) $-2uu_xu_y + u_x^2 = 0$

Add (i) and (ii) and we see $u_x^2 + u_y^2 = 0$ for all points in D. Hence $u_x = u_y = 0$ in D and u is constant. Therefore also v is constant and we see that $f = u + iv$ is constant.

(2) Let $f = u + iv$, where $u(z) = x^3 + axy^2$ and $v(z) = bx^2y + cy^3 + 1$. Determine the values of the <u>real</u> numbers $a, b,$ and c that make f an entire function. Then express f as a polynomial function of z. (<u>Hint</u>: ~~$x =$~~ Once you have determined $a, b,$ and c start think about how $z^n = (x+iy)^n$ looks. Here the binomial theorem is useful.)

<u>Solution</u>: We calculate $u_x = 3x^2 + ay^2$, $u_y = 2axy$, $v_x = 2bxy$, and $v_y = bx^2 + 3cy^2$.

The Cauchy-Riemann equations must hold for all x and y. So $u_x = v_y$ and $u_y = -v_x$ gives

$$\begin{cases} 3x^2 + ay^2 = bx^2 + 3cy^2 & \text{(i)} \\ 2axy = -2bxy & \text{(ii)} \end{cases}$$

for all x and y. Therefore $b = 3$ and $a = 3c$ from (i) and $a = -b$ from (ii).

That means $a = -3$, $b = 3$, and $c = -1$.

So $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + 1)$ is entire. Now $z^3 = (x+iy)^3 = x^3 + 3ix^2y + 3i^2y^2x + i^3y^3$

$$= x^3 - 3xy^2 + i(3x^2y - y^3)$$

So $f(z) = z^3 + i$.

(3) For each of the following functions determine the largest open set in which it is analytic, and compute its derivative in that set:

a) $f(z) = \dfrac{e^z - 1}{e^z + 1}$

b) $g(z) = \dfrac{\sin\sqrt{iz+1}}{z^2+1}$

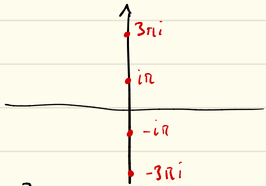
<u>Solution</u>: a) $f(z)$ is analytic if $e^z + 1 \neq 0$

We solve $e^z + 1 = 0 \iff e^z = -1 \iff$

$$z = \mathrm{Log}(-1) + i2\pi n = i\pi + i2\pi n =$$
$$= i(2n+1)\pi$$

Therefore $f(z)$ is analytic in the set

$$\mathbb{C} \setminus \{ z = i(2n+1)\pi \, ; \, n \in \mathbb{Z} \}$$

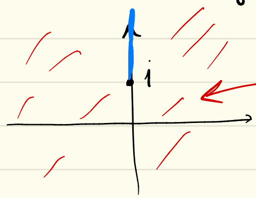


We have $f'(z) = \dfrac{e^z(e^z+1) - e^z(e^z-1)}{(e^z+1)^2} = \dfrac{2e^z}{(e^z+1)^2}$

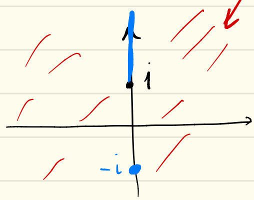
b) Remember that $\sqrt{\cdot}$ is the principal square root that is analytic in $\mathbb{C} \setminus \{x+i0;\ x \leq 0\}$. Therefore $\sin\sqrt{iz+1}$ is analytic outside the set where $\operatorname{Im}(iz+1)=0$ and $\operatorname{Re}(iz+1) \leq 0$.

Since $iz+1 = i(x+iy)+1 = (1-y)+ix$ we get that $\sin\sqrt{iz+1}$ is analytic outside the set $\{x+iy;\ x=0,\ 1-y \leq 0\} = \{x+iy,\ x=0, y \geq 1\}$

That is $\sin\sqrt{iz+1}$ is analytic here



For $g(z)$ to be analytic also $z^2+1 \neq 0$. Since $z^2+1=0$ iff $z = \pm i$ we get that $g(z)$ is analytic here



Since $\frac{d}{dz}\left(\sqrt{iz+1}\right) = \frac{i}{2\sqrt{iz+1}}$ we get

$$\frac{d}{dz} \sin\sqrt{iz+1} = \frac{i}{2\sqrt{iz+1}} \cos\sqrt{iz+1} \quad \text{and}$$

$$g'(z) = \frac{\frac{i\cos\sqrt{iz+1}}{2\sqrt{iz+1}} \cdot (z^2+1) - 2z\sin\sqrt{iz+1}}{(z^2+1)^2} =$$

$$= \frac{i(z^2+1)\cos\sqrt{iz+1} - 4z\sqrt{iz+1}\sin\sqrt{iz+1}}{2(z^2+1)^2\sqrt{iz+1}} .$$

(4) If a function f is analytic in a domain D and satisfies the differential equation

$$f'(z) - \alpha f(z) = 0 \text{ in } D \text{ for a}$$

constant $\alpha \in \mathbb{C}$ show that f takes the form $f(z) = Ae^{\alpha z}$ for some constant $A \in \mathbb{C}$.

(Hint: Study the function

$$g(z) = e^{-\alpha z} f(z). \quad)$$

Solution: The function $g(z) = e^{-\alpha z} f(z)$ is defined on D. Also

$$g'(z) = -\alpha e^{-\alpha z} f(z) + e^{-\alpha z} f'(z) =$$

$$= e^{-\alpha z} \underbrace{\left(f'(z) - \alpha f(z)\right)}_{=0} = 0$$

Since $g'(z) = 0$ on D we know that $g(z)$ is constant on D. That is $g(z) = A$ and

$$e^{-\alpha z} f(z) = A \quad \text{or} \quad f(z) = Ae^{\alpha z} .$$