Exercise sheet 6

(1) If $\gamma(t) = te^{it}$ for $0 \le t \le \pi$, calculate:

a) $\int_\gamma \bar{z}\, dz$

b) $\int_\gamma |z|\,|dz|$

c) $\int_\gamma z\, dz$

Solution: a) $\bar{z}$ is $\overline{\gamma(t)} = te^{-it}$ on $\gamma(t)$ and $\dot{\gamma}(t) = e^{it} + tie^{it} = (1+it)e^{it}$ so

$$\int_\gamma \bar{z}\, dz = \int_0^\pi (1+it)e^{it} \cdot te^{-it}\, dt = \int_0^\pi t + it^2\, dt =$$

$$= \left[\frac{t^2}{2} + i\frac{t^3}{3}\right]_0^\pi = \frac{\pi^2}{2} + i\frac{\pi^3}{3}$$

b) $|z|$ is $|\gamma(t)| = |t|$ on $\gamma(t)$ and $|\dot{\gamma}(t)| = |1+it| = \sqrt{1+t^2}$ so

$$\int_\gamma |z|\,|dz| = \int_0^\pi |t|\sqrt{1+t^2}\, dt = \begin{bmatrix} u = 1+t^2 \\ du = 2t\,dt \end{bmatrix} = \int_1^{1+\pi^2} \frac{1}{2}u^{1/2}\, du = \left[\frac{1}{3}u^{3/2}\right]_1^{1+\pi^2} = \frac{1}{3}\left((1+\pi^2)^{3/2} - 1\right)$$

c) $\int_\gamma z\, dz$ can be done in the same spirit.

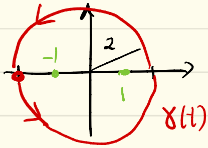
However, since $\frac{z^2}{2}$ is a primitive function of z we get

$$\int_\gamma z\, dz = \left[\frac{z^2}{2}\right]_0^{\pi e^{-i\pi}} = \frac{(-\pi)^2}{2} - 0 = \frac{\pi^2}{2}.$$

(2) Let $\gamma(t) = -2e^{it}$ for $0 \le t \le 2\pi$.

Evaluate $\displaystyle\int_\gamma \frac{1}{z^2-1}\,dz$.

Solution:



$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{A}{z-1} + \frac{B}{z+1} = \frac{(A+B)z+(A-B)}{(z+1)(z-1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = -1/2 \end{cases}$$

$$\int_\gamma \frac{1}{z^2-1}\,dz = \frac{1}{2}\int_\gamma \frac{1}{z-1}\,dz - \frac{1}{2}\int_\gamma \frac{1}{z+1}\,dz$$

We know that $\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$ in $\mathbb{C}\setminus(-\infty,0]$

Therefore, using $\gamma_\varepsilon(t) = \gamma(t)$ for $\varepsilon < t < 2\pi-\varepsilon$,

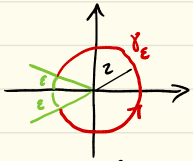
$$\int_\gamma \frac{1}{z^2-1}\,dz = \lim_{\varepsilon\to 0^+}\left(\frac{1}{2}\int_{\gamma_\varepsilon}\frac{1}{z-1}\,dz - \frac{1}{2}\int_{\gamma_\varepsilon}\frac{1}{z+1}\,dz\right) =$$

$$= \lim_{\varepsilon\to 0^+}\left(\frac{1}{2}\Big[\operatorname{Log}(z-1)\Big]_{\gamma(\varepsilon)}^{\gamma(2\pi-\varepsilon)} - \frac{1}{2}\Big[\operatorname{Log}(z+1)\Big]_{\gamma(\varepsilon)}^{\gamma(2\pi-\varepsilon)}\right)$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{2}\Big(i\operatorname{Arg}(-2e^{i(2\pi-\varepsilon)}-1) - i\operatorname{Arg}(-2e^{i\varepsilon}-1)$$

$$+ i\operatorname{Arg}(-2e^{i\varepsilon}+1) - i\operatorname{Arg}(-2e^{i(2\pi-\varepsilon)})\Big)$$

$$= \frac{1}{2}\big(i\pi - (-i\pi) + (-i\pi) - (i\pi)\big) = 0.$$



$$\left|\int_\gamma \frac{1}{z^2-1}\,dz - \int_{\gamma_\varepsilon}\frac{1}{z^2-1}\,dz\right| =$$

$$= \left|\int_{\text{green arc}} \frac{1}{z^2-1}\,dz\right| \le$$

$$\le M\cdot \ell(\text{green arc}) = 4M\varepsilon$$

M: max $\left|\frac{1}{z^2-1}\right|$ on green arc

Needed because $\operatorname{Log}(z)$ defined in $\mathbb{C}\setminus(-\infty,0]$

(3) Let a and b be real numbers satisfying $a<b$, and let $I(c)$ be defined for any real number c by

$$I(c) = \int_{c+ia}^{c+ib} e^{-z^2}\, dz.$$

Show that $\lim_{c\to\infty} |I(c)| = 0$ and $\lim_{c\to-\infty} |I(c)| = 0$.

Solution: We have $\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|$.

Using this and $\left|e^{-(c+it)^2}\right| \le e^{t^2-c^2}$ as $t\in[a,b]$. That is $\left|e^{-(c+it)^2}\right| \le e^{b^2-c^2}$ in $t\in[a,b]$.

Therefore $|I(c)| \le e^{b^2-c^2} \int_{c+ia}^{c+ib} |dz| =$

$$= e^{b^2-c^2}(b-a) = \frac{(b-a)e^{b^2}}{e^{c^2}}.$$

We get $\lim_{c\to\infty} |I(c)| \le \lim_{c\to\infty} \frac{(b-a)e^{b^2}}{e^{c^2}} = 0$

and $\lim_{c\to-\infty} |I(c)| \le \lim_{c\to-\infty} \frac{(b-a)e^{b^2}}{e^{c^2}} = 0$ ⊗

4 Evaluate the integrals (where $\gamma(t) = e^{it}, 0 \le t \le 2\pi$):

a) $\int_\gamma (z-2)^{-2}\,dz$

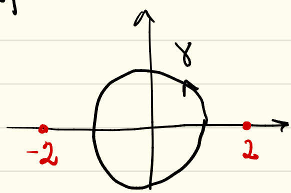
b) $\int_\gamma \frac{1}{z^2-4}\,dz$

c) $\int_\gamma \left(z+\frac{1}{z}\right)^n dz$ where n is a positive integer.

<u>Solution</u>: a) Notice that $\frac{d}{dz}\frac{1}{z-2} = -\frac{1}{(z-2)^2}$ and therefore $F(z) = -\frac{1}{z-2}$ is a primitive function of $\frac{1}{(z-2)^2}$ in $\mathbb{C}\setminus\{2\}$. Hence

$$\int_{|z|=1} \frac{1}{(z-2)^2}\,dz = 0.$$

b) We try to find a primitive function of $\frac{1}{z^2-4}$ in $\mathbb{C}\setminus\{2,-2\}$

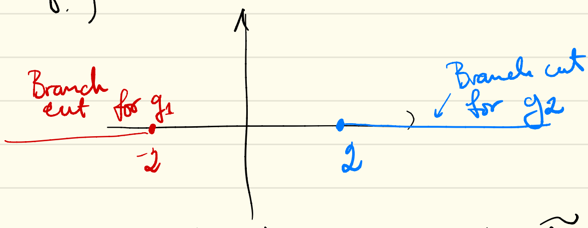


We use partial fractions decomposition

$$\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)} \overset{\text{Ansatz}}{=} \frac{A}{z-2} + \frac{B}{z+2} = \frac{Az+2A+Bz-2B}{(z-2)(z+2)}$$

$$\Longrightarrow \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Longrightarrow \begin{array}{l} A=\frac{1}{4} \\ B=-\frac{1}{4} \end{array}$$

$\frac{1}{z^2-4} = \frac{1/4}{z-2} - \frac{1/4}{z+2}$. We find a primitive function of $\frac{1}{z^2-4}$ use two branches of the logarithm. (We cannot use the principal branch since $\log(z-2)$ is not analytic on γ.)



$$g_1(z) = \mathrm{Log}(z+2) \quad \text{and} \quad g_2(z) = \widetilde{\log}(z-2)$$

where $\widetilde{\log}$ is a branch of the logarithm

$\widetilde{\log} : \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$. We get

$$\int_\gamma \frac{1}{z^2-4}\,dz = \left[\frac{1}{4} g_1(z)\right]_1^1 - \left[\frac{1}{4} g_2(z)\right]_1^1 = 0$$

c) First notice that $\int_\gamma z^k \, dz = 0$ if

$k \in \mathbb{Z} \setminus \{-1\}$ since $\frac{d}{dz}(z^{k+1}) = (k+1)z^k$

Also we know that $\int_\gamma \frac{1}{z} \, dz = 2\pi i$.

So, using the binomial theorem, we get

$$\left(z + \frac{1}{z}\right)^n = \left(z + z^{-1}\right)^n = \sum_{k=0}^{n} \binom{n}{k} z^k z^{k-n} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} z^{2k-n}$$

$\int_\gamma z^{2k-n} \, dz = 0$ unless $2k - n = -1$

or $k = \frac{n-1}{2}$

We have such a $0 \le k \le n$ if n is odd.

Therefore

$$\int_\gamma \left(z + \frac{1}{z}\right)^n dz = \begin{cases} 0 & n \text{ even} \\ 2\pi i \binom{n}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

We have

$$\binom{n}{\frac{n-1}{2}} = \frac{n!}{\left(\frac{n-1}{2}\right)! \left(n - \frac{n-1}{2}\right)!} = \frac{n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!}$$

So $$\int_\gamma \left(z + \frac{1}{z}\right)^n dz = \begin{cases} 0 & \text{if } n \text{ even} \\ 2\pi i \dfrac{n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} & \text{if } n \text{ odd} \end{cases}$$