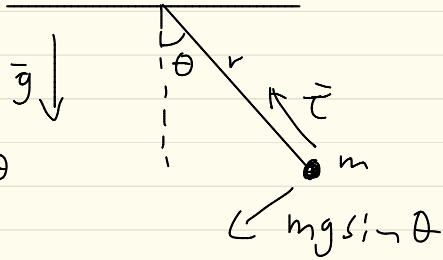
Some examples:

Pendulum



generalized coordinate θ

$\Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2$

$V = -mgl\cos\theta + \text{const}$

$\Rightarrow L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta - \text{const}$, $\dot{\theta}$ = generalized velocity

$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, $\frac{\partial L}{\partial q} = \frac{\partial L}{\partial \theta} = -mgl\sin\theta$

so $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = ml^2\ddot{\theta} + mgl\sin\theta = 0$

$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$

There was nothing special about θ btw. We could use

$q = \theta^{1/3} \Rightarrow \theta = q^3$, $\dot{\theta} = 3q^2\dot{q}$

$\Rightarrow L = \frac{1}{2}ml^2 9q^4\dot{q}^2 + mgl\cos q^3$

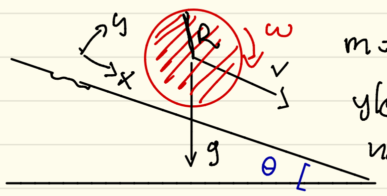
$\frac{\partial L}{\partial \dot{q}} = \frac{9}{2}ml^2q^4 2\dot{q} = 9ml^2q^4\dot{q}$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 9ml^2[4q^3\dot{q}^2 + q^4\ddot{q}]$

$\frac{\partial L}{\partial q} = 18ml^2\dot{q}^2q^3 - mgl(\sin q^3)\,3q^2$

$\Rightarrow 9ml^2q^4\ddot{q} + \underbrace{36ml^2q^3(\dot{q}^2 - \frac{1}{2}\dot{q}^2)}_{18ml^2q^3\dot{q}^2} + 3mgl\,q^2(\sin q^3) = 0$

$\rightarrow$ looks weird but is probably there.

Esimerkki: vierivä sylinteri



$m = \rho A = \rho \pi R^2$

yleistetty koordinaatti x

Painovoima $-mg\underbrace{x\sin\theta}_{x = \text{"hypotenuusa"}} = V(x)$

Massakeskipisteen liike-energia

$$T_1 = \frac{1}{2} m \dot{x}^2$$

Pyöriminen: $T_2 = \frac{1}{2} I \omega^2$, $\omega = v/R$ (= no-slip)

En muista I:tä $\Rightarrow$ lasken

pienen renkaan liike-energia

$$dT_2 = \frac{1}{2} dm(\omega r)^2, \quad dm = 2\pi r\, dr\, \rho$$

$$\Rightarrow T_2 = \int_0^R dr\, \frac{1}{2}\omega^2 \cdot 2\pi r^3 \rho = \frac{\omega^2 \rho \pi R^4}{4}$$

$$= \frac{1}{4}(mR^2)\omega^2, \quad \text{ts. } I = \frac{1}{2} m R^2$$

$$\Rightarrow T = T_1 + T_2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{4}(mR^2)\omega^2$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{4} m\dot{x}^2 = \frac{3}{4} m\dot{x}^2$$

$$L = \frac{3}{4} m\dot{x}^2 + mg\sin\theta\, x$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{3}{2} m\ddot{x} - mg\sin\theta = 0$$

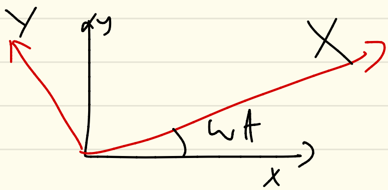
$$\Rightarrow \ddot{x} = \frac{2}{3} g \sin\theta$$

Example: rotating coordinate system?

You can move from one set of generalized coordinates to another quite easily. Rewrite the Lagrangian and then figure out equations of motions.

1) frame is stationary: $F=0 \Rightarrow \begin{matrix} m\ddot{x}=0 \\ m\ddot{y}=0 \end{matrix} \Rightarrow$ straight line

2) frame rotates. You stand on a carousel. What does the dynamics look to you?



$$x = X\cos\omega t + Y\sin\omega t$$
$$y = -X\sin\omega t + Y\cos\omega t$$

In system 1: $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$

In system 2: $L = ?$

$$\dot{x} = \dot{X}\cos\omega t - X\sin\omega t \cdot \omega + \dot{Y}\sin\omega t + Y\omega\cos\omega t$$
$$\dot{y} = -\dot{X}\sin\omega t - X\omega\cos\omega t + \dot{Y}\cos\omega t - Y\omega\sin\omega t$$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + \frac{m\omega}{2}(\dot{X}Y - \dot{Y}X)$$

Euler-Lagrange: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{X} + \frac{m\omega}{2}\dot{Y} - m\omega^2 X + \frac{m\omega}{2}\dot{Y}$

$= 0$

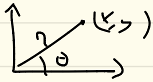
or $m\ddot{X} = m\omega^2 X - m\omega\dot{Y}$

centrifugal

Coriolis

likewise for $Y \Rightarrow m\ddot{Y} = m\omega^2 Y + m\omega\dot{X}$

Example: Polar coordinates



$L = \frac{m}{2}\left(\dot{r}^2 + (r\dot{\theta})^2\right) - V(r)$ (assume central force)

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

centrifugal

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Leftrightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

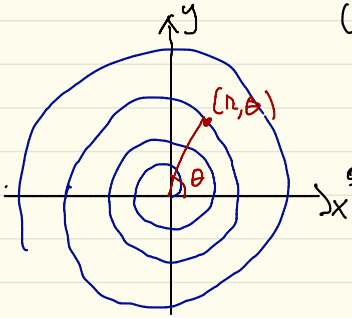
Constant of motion. It is angular momentum.

$$mr^2\dot{\theta} = L \Rightarrow \dot{\theta} = L/mr^2$$

Esimerkki: hiukkanen liikkuu spiraalilla.

Arkimedeen spiraali: $r = a\theta$

$$(x,y) = (r\cos\theta, r\sin\theta) = a(\theta\cos\theta, \theta\sin\theta)$$



$$\frac{dx}{dt} = \dot{x} = a\left[\dot\theta\cos\theta - \theta\dot\theta\sin\theta\right]$$

$$\dot{x}^2 = a^2\dot\theta^2\left[\cos^2\theta + \theta^2\sin^2\theta - 2\theta\cos\theta\sin\theta\right]$$

$$\dot{y} = a\left[\dot\theta\sin\theta + \dot\theta\theta\cos\theta\right]$$

$$\dot{y}^2 = a^2\dot\theta^2\left[\sin^2\theta + \theta^2\cos^2\theta + 2\theta\sin\theta\cos\theta\right]$$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{ma^2\dot\theta^2}{2}\left[1+\theta^2\right] \quad (\text{koska } \sin^2\theta + \cos^2\theta = 1)$$

Jos potentiaali esim. $V(r) = \frac{m\omega^2}{2}r^2 = \frac{m\omega^2 a^2}{2}\theta^2$

Lagrangen funktio: $L = T - V = \frac{ma^2}{2}\left[\dot\theta^2(1+\theta^2) - \omega^2\theta^2\right]$

Liikeyhtälö: $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0$, tässä siis $q = \theta$

$$\frac{\partial L}{\partial \theta} = ma^2\left[\dot\theta^2\theta - \omega^2\theta\right] = ma^2\theta\left[\dot\theta^2 - \omega^2\right]$$

$$\frac{\partial L}{\partial \dot\theta} = ma^2(1+\theta^2)\dot\theta, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = ma^2\ddot\theta(1+\theta^2) + 2ma^2\theta\dot\theta^2$$

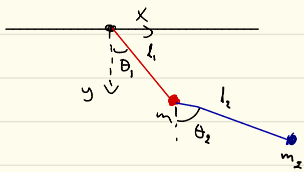
$$\Rightarrow ma^2\theta\left[\dot\theta^2 - \omega^2\right] - ma^2\ddot\theta(1+\theta^2) - 2ma^2\theta\dot\theta^2 = 0$$

$$\Rightarrow \ddot\theta(1+\theta^2) = \theta(-\omega^2 - \dot\theta^2)$$

$$\Rightarrow \ddot\theta = -\left(\frac{\omega^2 + \dot\theta^2}{1+\theta^2}\right)\theta$$

Numeerinen ratkaisu toisaalla.

2.3



1st pendulum : $x_1 = l_1 \sin\theta_1 \quad \dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1$

$y_1 = l_1 \cos\theta_1 \quad \dot{y}_1 = -l_1 \sin\theta_1 \dot{\theta}_1$

2nd pendulum: $x_2 = x_1 + l_2 \sin\theta_2 \quad \dot{x}_2 = \dot{x}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2$

$y_2 = y_1 + l_2 \cos\theta_2 \quad \dot{y}_2 = \dot{y}_1 - l_2 \sin\theta_2 \, \dot{\theta}_2$

$$T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2}\left[(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2\right]$$

potential energy : $V = -m g y_1 - m_2 g y_2 = -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$

$$L = T - V = L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$$

Equations of motion

$$\boxed{\begin{aligned} &1)\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0 \\ &2)\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0 \end{aligned}}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + \frac{m_2}{2}\left[2(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2) l_1 \cos\theta_1 + 2(l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2) l_1 \sin\theta_1\right]$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{m_2}{2}\left[2(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2) l_2 \cos\theta_2 + 2(l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2) l_2 \sin\theta_2\right]$$

$$\frac{\partial L}{\partial \theta_1} = m_2(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)(-l_1 \sin\theta_1 \dot{\theta}_1) + m_2(l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2) l_1 \cos\theta_1 \dot{\theta}_1$$
$$- m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \theta_2} = m_2(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)(-l_2 \sin\theta_2 \dot{\theta}_2) + m_2(l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2) l_2 \cos\theta_2 \dot{\theta}_2$$
$$- m_2 g l_2 \sin\theta_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1 \ddot{\theta}_1 + m_2 l_1^2(-2\cos\theta_1 \sin\theta_1 \dot{\theta}_1^2 + \cos^2\theta_1 \ddot{\theta}_1) + m_2 l_2 l_1(-\sin\theta_2 \cos\theta_1 \dot{\theta}_2^2 - \cos\theta_2 \sin\theta_1 \dot{\theta}_1 \dot{\theta}_2$$
$$+ \cos\theta_1 \cos\theta_2 \ddot{\theta}_2) + m_2 l_1(2\sin\theta_1 \cos\theta_1 \dot{\theta}_1^2 l_1 + l_1 \sin^2\theta_1 \ddot{\theta}_1 + l_2 \cos\theta_2 \sin\theta_1 \dot{\theta}_2^2 + \sin\theta_2 \cos\theta_1 \dot{\theta}_2 \dot{\theta}_1 + l_2 \sin\theta_2 \sin\theta_1 \ddot{\theta}_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2\left[-l_1 \sin\theta_1 \cos\theta_2 \dot{\theta}_1^2 - l_1 \cos\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 + l_1 \cos\theta_1 \cos\theta_2 \ddot{\theta}_1\right] + m_2 l_2\left[2\sin\theta_2 \cos\theta_2 \dot{\theta}_2^2 l_2 + l_2 \sin^2\theta_2 \ddot{\theta}_2\right]$$
$$+ m_2 l_2 l_1(\cos\theta_1 \sin\theta_2 \dot{\theta}_1^2 + \sin\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 + \sin\theta_1 \sin\theta_2 \ddot{\theta}_1) + m l_2^2\left[-2\cos\theta_2 \sin\theta_2 \dot{\theta}_2^2 + \cos^2\theta_2 \ddot{\theta}_2\right]$$

The ones with ——— cancel straight away.

Then you can use
$$\cos\theta_2 \sin\theta_1 - \sin\theta_2 \cos\theta_1 = \sin(\theta_1 - \theta_2)$$
$$\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1 = \sin(\quad)$$
$$\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = \cos(\theta_2 - \theta_1)$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = m_1 l_1^2 \ddot\theta_1 + m_2 l_1^2 \cos^2\theta_1 \ddot\theta_1 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1)\ddot\theta_2 - m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\dot\theta_2^2$$
$$+ m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\dot\theta_1 \dot\theta_2 + m_2 l_2^2 \sin^2\theta_1 \ddot\theta_1$$

$$\frac{\partial L}{\partial \theta_1} = +m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \underbrace{\left(-\cos\theta_2 \sin\theta_1 + \sin\theta_2 \cos\theta_1\right)}_{\sin(\theta_2 - \theta_1)} - (m_1 + m_2) g l_1 \sin\theta_1$$

$$\Rightarrow l_1^2 (m_1 + m_2)\ddot\theta_1 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1)\ddot\theta_2 - m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\dot\theta_2^2 + (m_1 + m_2) g l_1 \sin\theta_1 = 0 \quad \square$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1)\ddot\theta_1 + m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\left(\dot\theta_1^2 - \dot\theta_1 \dot\theta_2\right)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\dot\theta_1 \dot\theta_2 - g m_2 l_2 \sin\theta_2$$

$$\Rightarrow m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1)\ddot\theta_1 + g m_2 l_2 \sin\theta_2 + m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\dot\theta_1^2 = 0 \quad \square$$