Variational calculation

General problem: find a function, $y(x)$ say, for which a functional

$$I = \int_{x_1}^{x_2} \phi(y(x), y'(x), x)\, dx$$

is an extremum (often minimum)

what does this imply? Let $y(x)$ be the optimal choice and neighboring one is

$$Y(x) = y(x) + \varepsilon \eta(x) \quad (\varepsilon \text{ is small})$$

we keep end points fixed so $\eta(x_1) = \eta(x_2) = 0$
We don't care what $\eta(x)$ is. It is arbitrary.
We can compute the functional for this new path

$$I(\varepsilon) = \int_{x_1}^{x_2} \phi(Y(x), Y'(x), x)\, dx = \int_{x_1}^{x_2} \phi(y(x) + \varepsilon\eta(x), y' + \varepsilon\eta', x)\, dx$$

Expand the integrand in Taylor series about $\varepsilon = 0$

$$\Rightarrow I(\varepsilon) = \int_{x_1}^{x_2} \phi(y, y', x)\, dx + \varepsilon \int_{x_1}^{x_2} \left[\frac{\partial\phi}{\partial y} \eta(x) + \frac{\partial\phi}{\partial y'} \eta'(x) \right] dx + O(\varepsilon^2)$$

for extremum $\left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0$ and 2nd term should vanish

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial\phi}{\partial y} \eta(x) + \frac{\partial\phi}{\partial y'} \frac{d}{dx} \eta(x) \right] dx = 0,$$ use $\frac{\partial\phi}{\partial y'} \frac{d}{dx} \eta(x) = \frac{d}{dx}\left[\frac{\partial\phi}{\partial y'} \eta(x) \right] - \eta(x) \frac{d}{dx} \frac{\partial\phi}{\partial y'}$

1st term due to $\int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial \phi}{\partial y'}\eta(x)\right]dx = \left.\eta(x)\frac{\partial \phi}{\partial y'}\right|_{x_1}^{x_1} = 0$

since $\eta(x)$ vanishes at end points.

$$\Rightarrow \int_{x_1}^{x_2} \eta(x)\left[\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'}\right]dx = 0$$

This must be true for ANY $\eta(x)$ so square brackets must be zero

$$(*) \quad \boxed{\Rightarrow \frac{d}{dx}\frac{\partial \phi}{\partial y'} - \frac{\partial \phi}{\partial y} = 0 \quad \text{Euler-Lagrange equation}}$$

You can generalize this to more complicated integrands ϕ and for more than 1D.

Example: shortest distance between points? (Euclidian space)

$$S_{12} = \int_1^2 \sqrt{dx^2+dy^2} = \int_1^2 \sqrt{1+y'^2}\,dx \Rightarrow \phi = \sqrt{1+y'^2}$$

$$(*) \Rightarrow \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \Rightarrow y''\left\{\frac{1}{(1+y'^2)^{3/2}} - \frac{(y')^2}{(1+y'^2)^{3/2}}\right\} = 0$$

or $y'' = 0 \Rightarrow y(x) = ax+b$ amazing!

Let us do the derivation also in terms of variation $\delta y(x)$. This is same as above, but with more convenient notation.

$$Y(x) - y(x) = \varepsilon\eta(x) \equiv \delta y(x)$$

derivation $\Rightarrow Y'(x) - y'(x) = \varepsilon\eta'(x) \equiv \delta y'(x)$

$$\delta y'(x) = \frac{d}{dx}\delta y(x)$$

Variation of the functional:

$$\phi(Y(x), Y'(x), x) - \phi(y(x), y'(x), x) \equiv \delta\phi$$

Taylor series implies that

$$\delta\phi = \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta y' \quad \text{(up to 1st order)}$$

This then gives variation in I as

$$\delta I \equiv \int_1^2 \delta\phi\, dx = \int_1^2 \left(\frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta y'\right) dx = \int_1^2 \left(\frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\frac{d}{dx}\delta y\right) dx \quad (*)$$

fixed end points implies $\delta y(x_1) = \delta y(x_2) = 0$

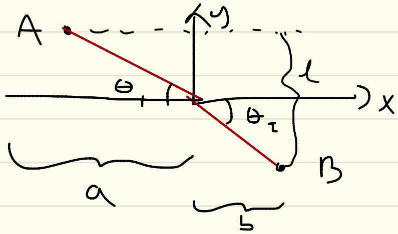
Partial integration of the 2nd term in $(*)$

$$\Rightarrow \delta I = \int_{x_1}^{x_2} \delta y \left[\frac{\partial\phi}{\partial y} - \frac{d}{dx}\frac{\partial\phi}{\partial y'}\right] dx$$

This must vanish for extremum $\Rightarrow$ **Euler-Lagrange equation**

Example: Fermat's principle

Light travels from A to B along such path that travel time is minimized



n_A = Index of refraction

$\rightarrow v_A = c/n_A$

$n_B \rightarrow v_B = c/n_B$

A at $(-a, y)$ & B at $(b, -(l-y))$

from A to origin: $d_A = \sqrt{a^2+y^2}$

from origin to B: $d_B = \sqrt{b^2+(l-y)^2}$

$$\begin{cases} T = \int_A^B ds/v_s \\ v_s = \begin{cases} c/n_A & \text{in } A \\ c/n_B & \text{in } B \end{cases} \end{cases}$$

$$T = \frac{d_A}{v_A} + \frac{d_B}{v_B} = \frac{\sqrt{a^2+y^2}}{v_A} + \frac{\sqrt{b^2+(l-y)^2}}{v_B}$$

$$\frac{dT}{dy} = \frac{y}{v_A\sqrt{a^2+y^2}} - \frac{(l-y)}{v_B\sqrt{(l-y)^2+b^2}} = 0$$

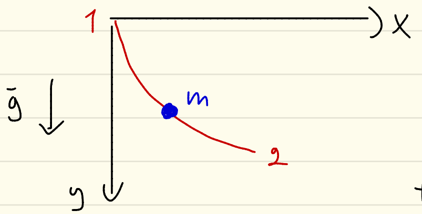
$$\Leftrightarrow \frac{\sin\theta_1}{v_A} - \frac{\sin\theta_2}{v_B} = 0 \Leftrightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

This is Snell's law.

Generally: $T = \int_{t_0}^{t_f} dt = \frac{1}{c}\int_{t_0}^{t_f} \frac{c}{v}\frac{ds}{dt}\,dt = \frac{1}{c}\int_A^B n\,ds$

$ds = \frac{\sqrt{dx^2+dy^2}}{dx}dx = \sqrt{1+(dy/dx)^2}\,dx$

Example: Brachistochrone-problem



What path minimizes the travel time from 1 to 2?

$$t_{12} = \int_1^2 dt = \int_1^2 \frac{ds}{v}$$

Energy conservation: $\frac{1}{2}mv^2 = mgy$

$$\Rightarrow t_{12} = \int_1^2 \underbrace{\frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}}_{\phi(y,\,y',\,x)} dx \quad , \text{ since } ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+(y')^2}$$

x doesn't appear in ϕ explicitly $\Rightarrow$ we can use so-called Beltrami identity. In Euler-Lagrange eq. we calculate

$$\frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right) = \frac{\partial^2 \phi}{\partial y \partial y'} y' + \frac{\partial^2 \phi}{\partial y'^2} y'' \quad \text{(Chain rule)}$$

E-L eq. $\Rightarrow \dfrac{\partial^2 \phi}{\partial y \partial y'} y' + \dfrac{\partial^2 \phi}{\partial y'^2} y'' = \dfrac{\partial \phi}{\partial y}$

Calculate

$$\frac{d}{dx}\left(y'\frac{\partial \phi}{\partial y'} - \phi\right) = \left(\frac{\partial \phi}{\partial y'} y'' + \frac{\partial^2 \phi}{\partial y \partial y'} y'^2 + \frac{\partial^2 \phi}{\partial y'^2} y' y''\right) - \left(\frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial y'} y''\right)$$

$$= y' \underbrace{\left(\frac{\partial^2 \phi}{\partial y \partial y'} y' + \frac{\partial^2 \phi}{\partial y'^2} y'' - \frac{\partial \phi}{\partial y}\right)}_{0} = 0$$

$$\Leftrightarrow y'\frac{\partial \phi}{\partial y'} - \phi = C = \text{constant}$$

Brachistochrone continues ...

Ignore $2g$ since it is just a constant prefactor

$$\phi(y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}} \quad , \quad y'\frac{\partial\phi}{\partial y'} - \phi = C$$

$$\Rightarrow \frac{y'^2}{\sqrt{(1+y'^2)}\sqrt{y}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = C \Leftrightarrow \frac{-1}{\sqrt{1+y'^2}\sqrt{y}} = C$$

Solve for y' $\Rightarrow \frac{dy}{dx} = \sqrt{\frac{k-y}{y}}$, $k = 1/C^2$

This can be a bit tricky, but $\tan\theta = \sqrt{\frac{y}{k-y}} \Rightarrow y = k\sin^2\theta$

$$\frac{d\theta}{dx} = \frac{d\theta}{dy}\frac{dy}{dx} = \frac{1}{2k\sin\theta\cos\theta}\cdot\frac{1}{\tan\theta} = \frac{1}{2k\sin^2\theta}$$

so $dx = 2k\sin\theta\, d\theta \Rightarrow$ integrate $\Rightarrow x = 2k\int\sin^2\theta\, d\theta$

$$= 2k\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C_1$$

Initial conditions: start at $(x,y) = (0,0)$

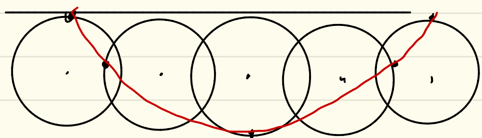
$y = k\sin^2\theta = 0 \Rightarrow \theta = 0$. substitute to eq. for x

$\Rightarrow x = C_1$ or $C_1 = 0$

Define $a = k/2$, $2\theta = \alpha$

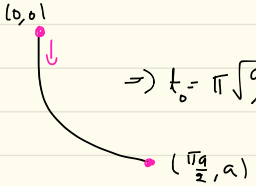
$$\Rightarrow \begin{cases} x = a(\alpha - \sin\alpha) \\ y = a(1-\cos\alpha) \end{cases}$$

Cycloid



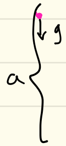
Curve traced out by a fixed point on a circle of radius a rolling in a straight line.

Time it takes to fall: integrate the starting point using the found solution.



$\Rightarrow t_0 = \pi\sqrt{a/2g}$

Free fall over a :

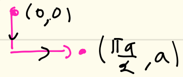


$y = \frac{1}{2} g t^2$

$\Rightarrow t_F = \sqrt{\frac{2a}{g}}$

$t_0/t_F = \frac{\pi}{\sqrt{2}\sqrt{2}} = \frac{\pi}{2} > 1$ so it takes longer than free fall. Makes sense.

Final point $(\frac{\pi a}{2}, a)$: alternative guess. Drop straight $\rightarrow$ turn right and travel at fixed vel.



$\rightarrow t_A = t_F + \frac{\pi a}{2v}$, $\frac{1}{2} m v^2 = mga$, $v = \sqrt{2ga}$

$$\Rightarrow t_A = \sqrt{\frac{2a}{g}} + \frac{\pi a}{2\sqrt{2ga}} = \sqrt{\frac{a}{g}}\left[\sqrt{2} + \frac{\pi}{2\sqrt{2}}\right] = \pi\sqrt{\frac{a}{2g}}\underbrace{\left[\frac{2}{\pi} + \frac{1}{2}\right]}_{\approx 1.14}$$

$\Rightarrow$ so we have certainly found a faster path than this!

Esimerkki Suspended cable:

call linear density in the cable as ρ. Length of the cable L.

Potential energy of the cable: (there is no kinetic energy)

$$V = \rho g \int_0^L y\,ds = \rho g \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

There is a constraint in that lenght must be L.

$$\Rightarrow \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = L$$

We are looking for optimum with integrand

$$F = \rho g y\sqrt{1+y'^2} - \lambda\sqrt{1+y'^2} = (\rho g y - \lambda)\sqrt{1+y'^2}$$

there is no explicit x-dependence so we can use Beltrami identity (amounts to Euler-Lagrange)

$$y'\frac{\partial F}{\partial y'} - F = C = \text{constant}$$

$$\Rightarrow \frac{y'^2}{\sqrt{1+y'^2}}(\rho g y - \lambda) - (\rho g y - \lambda)\sqrt{1+y'^2} = C$$

which simplifies to $-\dfrac{\rho g y - \lambda}{\sqrt{1+y'^2}} = C$

solve for $y' \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{\rho g y - \lambda}{C}\right)^2 - 1$

$\rho g y - \lambda = Cz$ change of variables

$\rho g\,dy = C\,dz$

(3.4) ... continues

$$\Rightarrow \frac{dz}{\sqrt{z^2-1}} = \frac{\rho g}{c}\,dx \Rightarrow \left(\ln\left(z+\sqrt{z^2-1}\right)\right) = \frac{\rho g}{c}x + b$$ ← constant of integration

$b = -\frac{\rho g}{c}\beta$ to simplify algebra

$$\Rightarrow z = \frac{\rho g y - \lambda}{c} = \cosh\left[\frac{\rho g(x-\beta)}{c}\right]$$

$$\Rightarrow y(x) = +\frac{\lambda}{\rho g} + \frac{c}{\rho g}\cosh\left[\frac{\rho g(x-\beta)}{c}\right]$$, we can choose $\beta = 0$ so cable hangs around $x = 0$. Then $x_1 = -x_2$

$y(x_2) = h = y(x_1)$

Length $L = \int_{-x_2}^{x_2} \sqrt{1+y'^2}\,dx = \int_{-x_2}^{x_2} \cosh\frac{\rho g x}{c}\,dx = \frac{2c}{\rho g}\sinh\left(\frac{\rho g x_2}{c}\right)$

If we know ρ, g, x_2 this determines c after which boundary condition $y(x_2) = h$ sets λ.

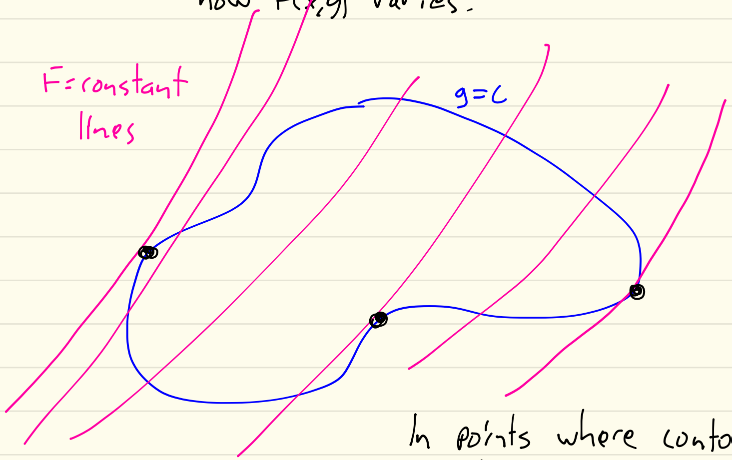
This shape is known as a catenary.

why did we apply Euler-Lagrange to

$$M = \int (\phi - \lambda g)\, dx \ ??$$

Plausibility: Say you want extrema of $F(x,y)$ with constraint $g(x,y) = C$.

Strategy: Walk along the contour $g(x,y) = C$ and see how $F(x,y)$ varies.



In points where contours have same tangent F changes in the same direction on either side $\Rightarrow F =$ extremum

Mathematically: $\nabla F \parallel \nabla g \iff \nabla F = \lambda \nabla g$ for some λ

$$\Rightarrow \begin{cases} \nabla F = \lambda \nabla g \\ g(x,y) = C \end{cases}$$ pair of equations

$\Rightarrow \nabla(F - \lambda g) = 0 \Rightarrow$ looking for where $F - \lambda g$ has extrema.