Hamilton's principle

Euler-Lagrange equations from the previous optimization look very similar to Lagrange equations for generalized coordinates. ⇒ Variational basis??

Define action as $S = \int_{t_1}^{t_2} dt\, L(q_1 \ldots q_n; \dot{q}_1 \ldots \dot{q}_n, t)$

Make variation for each generalized coordinate

$$\delta q_\sigma(t) = \varepsilon \eta_\sigma(t), \quad \sigma = 1, \ldots n$$

Hamilton's principle states: $\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0$

⇒ based on earlier $\int_{t_1}^{t_2} dt \sum_{\sigma=1}^{n} \left(\frac{\partial L}{\partial q_\sigma} \delta q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma} \delta \dot{q}_\sigma \right) = 0$

$\delta q_\sigma(t_1) = \delta q_\sigma(t_2) = 0 \Rightarrow \int_{t_1}^{t_2} dt \left[\sum_\sigma \delta q_\sigma \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \right] = 0$

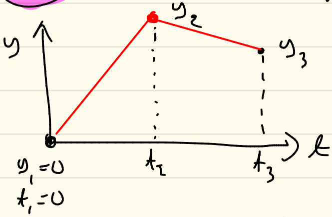
$$\boxed{\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \iff \text{Lagrange equations}}$$

Note: this principle is also very often called as principle of least action.

Or in the old days (according to Feynman)

"the principle of least $\underbrace{\text{Hamilton's first principal function}}_{= \text{action} = \int dt\, L = S}$"

Hamilton's principle: example



$dt = t_{i+1} - t_i$

gravity: $V = +mg\,y(t)$

$K = \frac{1}{2} m \dot{y}^2$

$$S = \int dt\, L = \int dt\,(K-V) \approx dt\left[\frac{L_1+L_2}{2} + \frac{L_3+L_2}{2}\right]$$

$$\frac{L_1+L_2}{2} = \frac{1}{2} m\left(\frac{y_2-y_1}{dt}\right)^2 - mg\,\frac{(y_2+y_1)}{2}, \quad \frac{L_2+L_3}{2} = \frac{1}{2} m\left(\frac{y_3-y_2}{dt}\right)^2 - mg\,\frac{y_3+y_2}{2}$$

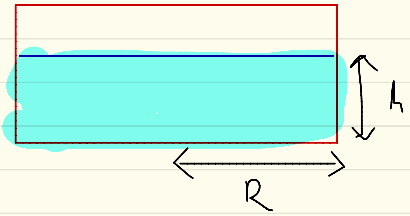
$$\frac{dS}{dy_2} = m\,\frac{y_2-y_1}{dt^2} - m\,\frac{y_3-y_2}{dt^2} - \frac{mg}{2} - \frac{mg}{2} = 0$$

$$\Leftrightarrow -m\,\frac{(y_1-2y_2+y_3)}{dt^2} - mg = 0 \qquad \left(\Rightarrow m\,\frac{d^2y}{dt^2} = -mg \rightarrow y = -\frac{1}{2} g t^2 + v_0 t + y_0\right)$$

$$\Rightarrow y_2 = -\frac{mg\,dt^2}{2m} + \frac{y_1+y_3}{2} = -\frac{1}{2} g\,dt^2 + \frac{y_1+y_3}{2}$$

Note: You could also guess a solution $y(t)$ in terms of some trial function. Then compute S and optimize to find parameters of the trial function.

Hamilton's principle: liquid mirror example



Rotate this cylinder with liquid at angular velocity ω.

What form does the surface take?

Cylindrical coordinates: (r, θ, z) so that

$$(x, y) = (r\cos\theta,\ r\sin\theta)$$

Surface of liquid some function $z = f(r)$

Element around (r, θ, z) has volume $dV = r\,dr\,d\theta\,dz$

Let us keep density constant at ρ (incompressible)

$\Rightarrow dm = \rho r\,dr\,d\theta\,dz$ & pot. energy $dV_g = \rho g r\,dr\,dz\,d\theta\,z$

Total potential energy is

$$V_g = \int dV = \rho g \int_0^{2\pi} d\theta \int_0^R \left(\int_0^{f(r)} z\,dz \right) r\,dr = 2\rho g \pi \int_0^R \Big/_0^{f(r)} \frac{z^2}{2}$$

$$= \rho g \pi \int_0^R (f(r))^2 r\,dr$$

Then kinetic energy: $dT = \frac{1}{2} v^2 dm$, $v = r\omega$

$$\Rightarrow T = \int dT = \frac{1}{2}\rho\omega^2 \int_0^{2\pi} d\theta \int_0^R \int_0^{f(r)} dz\, r^3\,dr = \rho\pi\omega^2 \int_0^R f(r)\, r^3\,dr$$

$$L = T - V = \rho\pi \int_0^R \underbrace{\left(f(r)\omega^2 r^3 - g\,(f(r))^2 r \right)}_{\phi(f, r)} dr$$

Rotating liquid mirror continues..

There is a constraint, more of these later, but I will now show first example how they enter the problem.

Mass cannot change and since density is constant so is volume.

$$\Rightarrow \text{Volume} = \int_0^{2\pi} d\theta \int_0^R \int_0^{f(r)} dz \, r\,dr = 2\pi \int_0^R r f(r)\, dr$$

Under constraint $J = \int_{x_1}^{x_2} g(y, y', x)\, dx$ the optimization of $I = \int_{x_1}^{x_2} \phi(y, y', x)\, dx$ amounts to finding a solution to the Euler-Lagrange equation with the functional

$$M = \int_{x_1}^{x_2} \left(\phi(y, y', x) - \lambda g(y, y', x) \right) dx$$

So in Hamilton's principle we have to replace Lagrangian L with $L \to L - \lambda g$ to account for the constraint.

Here this means:

$$\phi \to \rho \omega^2 f(r) r^3 - \rho g (f(r))^2 r - 2\lambda r f(r)$$

(note: I dropped out irrelevant π here)

Euler-Lagrange: $\frac{\partial \phi}{\partial f} - \frac{d}{dr} \frac{\partial \phi}{\partial f'} = 0 \Leftrightarrow \frac{\partial \phi}{\partial f} = 0$

$$\Rightarrow \rho \omega^2 r^3 - 2\rho g f(r) r - 2\lambda r = 0 \quad \text{so} \quad f(r) = \frac{\omega^2 r^2}{2g} - \frac{\lambda}{\rho g}$$

PARABOLA

λ determined from constraint

$$\text{Volume} = 2\pi \int_0^R \left(\frac{\omega^2 r^2}{2g} - \frac{\lambda}{\rho g} \right) r\,dr = \frac{\pi \omega^2}{4g} R^4 - \frac{\pi \lambda R^2}{\rho g} = \underbrace{\pi R^2 h}_{\text{without rotation}}$$

$$\Rightarrow \lambda = \frac{\rho \omega^2 R^2}{4} - \rho g h$$

Liquid mirror continues...

For a parabola $z = ar^2$ the focal length btw is

$f = \frac{1}{4a}$ so in our example it is

$$f = \frac{1}{4}\frac{2g}{\omega^2} = \frac{1}{2}\frac{g}{\omega^2}$$

Forces of constraint:

Hamiltons principle using generalized coordinates $\{q_\sigma\}$

$$\int_{t_1}^{t_2} \left[\sum_\sigma \delta q_\sigma \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \right] dt = 0$$

Here we incorporate holonomic constraints using Lagrange multipliers. Why? Wasn't the point of earlier formulations to write things in such a way, that forces of constraint are not needed?
Sometimes you might need to know what some of those constraining forces are. If k constraints, at one extreme 3N-k independent generalized coordinates and no constraining forces. In the other 3N coordinates + constraints.

Constraints: $f_j(q_1 \dots q_n, t) = c_j \quad j = 1 \dots k$

Variation of these: $\delta f_j = \sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma = 0 \quad j = 1 \dots k$

Multiply these with some $\lambda_j = \lambda_j(q_1 \dots q_n)$, RHS of course zero

$$\Rightarrow \boxed{\int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \delta q_\sigma \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} \right) dt = 0}$$

(we just added zero here)

Independent variations: $\sigma = 1 \dots n-k$
Non-independent variations: $\sigma = n-k+1 \dots n$

We can choose functions $\lambda_1 \dots \lambda_k$ so that coefficients of $\delta q_{n-k+1} \dots \delta q_n$ vanish identically.

... Hamilton's principle + constraints

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q_\sigma} - \dfrac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \dfrac{\partial f_j}{\partial q_\sigma} \quad , \sigma = 1 \ldots n \\ f_j(q_1, \ldots q_n, t) = C_j \quad , j = 1 \ldots k \end{array}} \qquad (*)$$

$n+k$ equations & $n+k$ unknowns $(q_1, \ldots q_n, \lambda_1 \ldots \lambda_k)$

Lagrange multipliers determine reaction forces.

Remember that earlier, before we introduced potential for conservative forces, we had Lagrange equation in a form.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma \quad \sigma = 1 \ldots n$$

generalized force : work done $\delta W = \sum_\sigma Q_\sigma \delta q_\sigma$

$$L = T - V + (*) \Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot q_\sigma} - \frac{\partial T}{\partial q_\sigma} = \underbrace{-\frac{\partial V}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma}}_{Q_\sigma} \qquad \sigma = 1 \ldots n$$

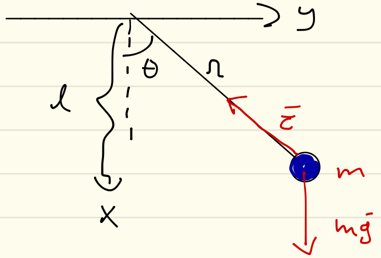
Generalized force has two contributions.

1) $-\frac{\partial V}{\partial q_\sigma}$ force from the potential V

2) $Q_\sigma^n = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma}$ from forces of constraint and not included in lagrangian for unconstrained motion. Generalized forces that are exerted by constraints to force constrained trajectories. $\delta W^n = \sum_\sigma Q_\sigma^n \delta q_\sigma$

Interpretation of Q_σ^n as reaction forces in general requires independent variations δq_σ that violate the constraint. For example, if string has length $r = \ell$. Then to identify Q_r means carrying out virtual displacement in r. $\delta W = Q_r \delta r = -\tau \delta r \rightarrow Q_r = -$tension

Forces of constraint: Pendulum example



Generalized coordinates: r, θ
What is the force of constraint associated with coordinate r?

$$x = r\cos\theta \qquad y = r\sin\theta \Rightarrow L = T - V = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + mgr\cos\theta$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

Constraint: $r = l \qquad f(r,\theta,t) = l$ $(*)$, one constraint so $k=1$

$$\delta r = 0 \qquad \partial f/\partial r = 1, \quad \frac{\partial f}{\partial \theta} = 0$$

Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial \theta} = Q_\theta = 0 \qquad (**)$$

$$(***) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial r} = \lambda = Q_r = \text{generalized reaction force for } r.$$

Note: since there was only one constraint $\lambda_j \to \lambda$

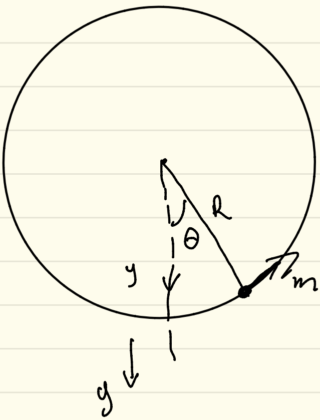
Virtual work done under displacement δr

$$\delta W = Q_r \delta r = -\tau\,\delta r \qquad \tau = \text{tension}$$
$$= \lambda\,\delta r$$

(note that δr breaks the constraint. We broke the constraint to have a physical understanding of the reaction force.)

1.3.

Lagrangen formalismi + tukivoimat?



$T = \frac{m}{2}(r^2\dot{\theta}^2 + \dot{r}^2)$, $V = mg\cos\theta\, r$

$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mg\cos\theta\, r$

rajoite: $q_1 = \theta$, $q_2 = r$

$f_1(\theta, r) = r - R = 0$ (1)

(ainut rajoite)

$\delta r = 0$, $\partial f_1/\partial r = 1$, $\partial f_1/\partial \theta = 0$

Lagrangen yhtälöt: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial f_1}{\partial \theta} = 0$ (2)

$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda_1 \frac{\partial f_1}{\partial r} = \lambda_1$ (3)

(3) $\Rightarrow -\frac{m}{2}\cdot 2r\dot{\theta}^2 - mg\cos\theta = \lambda_1$ (koska $\dot{r} = \ddot{r} = 0$ (1):n perusteella)

$\Rightarrow \lambda_1 = -mg\cos\theta - mr\dot{\theta}^2 = -mg\cos\theta - \frac{mv(\theta)^2}{r}$ = − jännite langassa

Energia säilyy: Jos nopeus $r\dot{\theta}|_{\theta=0} = v$

$\underbrace{\frac{m}{2}v^2 - \frac{m}{2}v(\theta)^2}_{\text{liike-energian muutos}} = \underbrace{mgR(1-\cos\theta)}_{\text{potentiaalienergian muutos}} \Rightarrow v(\theta)^2 = v^2 - 2gR(1-\cos\theta)$

Tällöin $\lambda_1 = -\frac{m}{R}(v^2 - 2gR(1-\cos\theta)) - mg\cos\theta = -\frac{mv^2}{R} + 2mg - 3mg\cos\theta$

negatiivinen, kun $\theta = 0$

Lanka löystyy jos $\lambda_1 > 0$: $\theta = \pi \Rightarrow \lambda_1 = -\frac{mv^2}{R} + 5mg < 0 \Rightarrow v^2 > \sqrt{5gR}$

How does the particle know? Isn't Hamilton's principle somehow magic?

$$S = \int_{t_0}^{t_f} dt\, L(x, \dot{x}) \to \varepsilon \sum_i L(x_i, \dot{x}_i)$$

$t_{i+1} - t_i = \varepsilon$

On the other hand:

$$\dot{x}_i = \frac{x_{i+1} - x_i}{\varepsilon}$$

x_i appears in the sum in terms "i" & "$i-1$"
we look for an extremum in x_i and keep other x_is fixed. So relevant term

$$S_i = \varepsilon \left[L\Big(x_{i-1}, \underbrace{\frac{x_i - x_{i-1}}{\varepsilon}}_{v_i}\Big) + L\Big(x_i, \frac{x_{i+1} - x_i}{\varepsilon}\Big) \right]$$

Vary x_i:

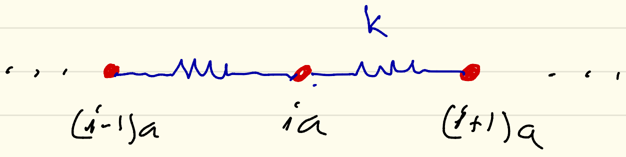
$$dS_i = \varepsilon \left[\frac{d\, L(x_i, v_i)}{dx_i} + \frac{\partial L(x_i, v_i)}{\partial v_i} \cdot \frac{1}{\varepsilon} - \frac{\partial L(x_i, v_{i+1})}{\partial v_{i+1}} \frac{1}{\varepsilon} \right] dx_i$$

$$= \varepsilon \left[-\frac{d}{dt} \frac{\partial L}{\partial v_i} + \frac{dL}{dx_i} \right]$$

$$dS_i = 0 \Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0}$$ i.e. Lagrange's equation

So the particle doesn't have to "know" about anything beyond its immediate position. Differential eqs. are consistent with "global" Hamilton's principle.

Kuinka siirrytään hiukkasista kenttiin? Esimerkki



Tasapainopaikat: $\bar{x}_n = na$

Paikat tasapainon lähellä:

$x_n(t) = \bar{x}_n + \phi_n(t)$

Potentiaali: $V = \sum_{n=1}^{N} \frac{k}{2}(x_{n+1} - x_n - a)^2 = \sum_{n=1}^{N} \frac{k}{2}\left(\phi_{n+1}(t) - \phi_n(t) + (n+1)a - na - a\right)^2$

kaikkien hiukkasten yli summa

$= \sum_{n=1}^{N} \frac{k}{2}\left(\phi_{n+1}(t) - \phi_n(t)\right)^2$, vaikka periodinen reunaehto

$\Rightarrow x_{N+1} = Na + x_1$

Jatkumoraja: $\phi_n \to a^{1/2}\phi(x)\big|_{x=na}$, $\phi_{n+1} - \phi_n \to a^{3/2}\partial_x\phi(x)\big|_{x=na}$

$$\sum_{n=1}^{N} \to \frac{1}{a}\int_0^L dx$$

Liike-energia siis: $T = \sum_{n=1}^{N} \frac{m}{2}\dot{x}_n^2 = \sum_{n=1}^{N} \frac{m}{2}\dot{\phi}_n(t)^2$

$\Rightarrow L[\phi] = \int_0^L dx\, \mathcal{L}(\dot{\phi}, \partial_x\phi)$, missä Lagrangen tiheys

$$\mathcal{L}(\dot{\phi}, \partial_x\phi) = \frac{m}{2}\dot{\phi}^2 - \frac{ka^2}{2}(\partial_x\phi)^2$$

Vaikutus $S[\phi] = \int dt \int_0^L dx\, \mathcal{L}(\dot{\phi}, \partial_x\phi)$

Sovelletaan nyt Hamiltonin periaatetta tähän vaikutukseen.

eli tehdään korvaus $\phi(x,t) \longrightarrow \phi(x,t) + \varepsilon\,\eta(x,t)$

pieni   joku variaatio

ja vaaditaan, että ensimmäinen kertaluku ε termissä häviää.

$$\Rightarrow S[\phi+\varepsilon\eta] = \iint dt\,dx\ \frac{m}{2}\left[\frac{\partial}{\partial t}(\phi+\varepsilon\eta)\right]^2 - \frac{Ka^2}{2}\left[\frac{\partial}{\partial x}(\phi+\varepsilon\eta)\right]^2$$

$$\approx \iint \frac{m}{2}\left[\dot\phi^2 + 2\varepsilon\dot\phi\dot\eta\right] - \frac{Ka^2}{2}\left[(\partial_x\phi)^2 + 2\varepsilon\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}\right]$$

$$= S[\phi] + 2\varepsilon\iint \frac{m}{2}\dot\phi\dot\eta - \frac{Ka^2}{2}\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}$$

Haluamme integrandin JOTAIN $\times\ \eta(x,t)$, josta JOTAIN $=0$

$\Rightarrow$ osittaisintegrointi ajan suhteen ensimmäisessä termissä

— " — paikan — " — toisessa — " —

JA $\eta(0,t) = \eta(L,t) = \eta(x,0) = \eta(x,T) = 0$

$$\Rightarrow S[\phi] + 2\varepsilon\iint -\frac{m}{2}\ddot\phi\,\eta(x,t) + \frac{Ka^2}{2}\frac{\partial^2\phi}{\partial x^2}\eta(x,t)$$

Kun vaaditaan, että toinen termi on nolla mielivaltaisella variaatiolla $\eta(x,t)$

$$\Rightarrow m\frac{\partial^2\phi}{\partial t^2} - Ka^2\frac{\partial^2\phi}{\partial x^2} = 0$$

eli aaltoyhtälö (pitkittäisille aalloille ketjussa)

Normaali diffis korvautui osittaisdifferentiaaliyhtälöllä.