# Generalized momentum, cyclic coordinates:

Conservative holonomic system $\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$, $n-k$ independent coordinates $q_\sigma$

Define: $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$ = generalized momentum or canonical momentum

$$\Rightarrow \dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma}$$

If $q_\sigma$ doesn't appear explicitly in $L$ it is called cyclic.
then $\partial L / \partial q_\sigma = 0$

$$\Rightarrow \dot{p}_\sigma = 0 \text{ and } p_\sigma = \text{constant of motion.}$$

Note: $p_\sigma$ is not always linear momentum. Use the above definition to compute it.

Conservation laws related to symmetries. In case of a cyclic coordinate you can change the coordinate $q_\sigma \rightarrow q_\sigma + \varepsilon$ and lagrangian stays the same.

# 1. Hamiltonian dynamics

Previously: $\frac{d}{dt}\frac{\partial L}{\partial \dot q_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$ & $p_\sigma = \frac{\partial L}{\partial \dot q}$

$$\Rightarrow \frac{d}{dt} p_\sigma = \frac{\partial L}{\partial q_\sigma}$$

Lagrangian was $L = T - V$

$$\frac{d}{dt} L = ? = \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q = \dot p \dot q + p \ddot q = \frac{d}{dt}(p\dot q) \neq 0$$

There is no "conservation law of the lagrangian"

<u>But</u> define Hamiltonian $H = \sum_\sigma p_\sigma \dot q_\sigma - L$

(1) then $dH = \sum_\sigma \left( p_\sigma d\dot q_\sigma + \dot q_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \underbrace{\frac{\partial L}{\partial \dot q_\sigma}}_{p_\sigma} d\dot q_\sigma \right) - \frac{\partial L}{\partial t} dt$

these cancel

$$\Rightarrow dH = \sum \left( \dot q_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma \right)$$

$$\Rightarrow \frac{dH}{dt} = \sum_\sigma \left( \dot q_\sigma \dot p_\sigma - \frac{\partial L}{\partial q_\sigma} \dot q_\sigma \right) - \frac{\partial L}{\partial t}$$ on the other hand $\dot p_\sigma = \frac{\partial L}{\partial q_\sigma}$

$$\Rightarrow \frac{dH}{dt} = 0 - \frac{\partial L}{\partial t}$$ $\Rightarrow$ If $L$ has no explicit time dependence $H = \text{constant}$

(2.) If potential $V(x_1, .., x_n)$ doesn't depend on time and constraints are time independent, then $H$ is the total energy of the system. Why?

$$\dot x_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot q_\sigma + \underbrace{\frac{\partial x_i}{\partial t}}_{0} = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot q_\sigma$$

Kinetic energy is then $T = \frac{1}{2}\sum_i m_i \dot{x}_i^2 = \frac{1}{2}\sum_\sigma \sum_\lambda \underbrace{\left(\sum_i m_i \frac{\partial x_i}{\partial q_\sigma}\frac{\partial x_i}{\partial q_\lambda}\right)}_{\equiv m_{\sigma\lambda}} \dot{q}_\sigma \dot{q}_\lambda$

$$m_{\sigma\lambda} = m_{\lambda\sigma} = m_{\sigma\lambda}^{*} = m_{\lambda\sigma}^{*}$$

$\Rightarrow m_{\sigma\lambda}$ is real and symmetric matrix

$T = \frac{1}{2}\sum_\sigma \sum_\lambda m_{\sigma\lambda}(q)\, \dot{q}_\sigma \dot{q}_\lambda$ (quadratic form btw)

Then $H = \sum_\sigma p_\sigma \dot{q}_\sigma - L = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L = \sum_\sigma \frac{\partial (T-V)}{\partial \dot{q}_\sigma}\dot{q}_\sigma - (T-V) = \sum_\sigma \underbrace{\frac{\partial T}{\partial \dot{q}_\sigma}}_{(*)}\dot{q}_\sigma - T + V$

where we assumed potential that didn't depend on generalized vecolicities.

$(*) = \frac{\partial}{\partial \dot{q}_\sigma}\frac{1}{2}\sum_\mu \sum_\lambda m_{\mu\lambda}\dot{q}_\mu \dot{q}_\lambda = \frac{1}{2}\sum_\lambda m_{\sigma\lambda}\dot{q}_\lambda + \frac{1}{2}\sum_\mu \underbrace{m_{\mu\sigma}}_{\equiv m_{\sigma\mu}}\dot{q}_\mu = \sum_\lambda m_{\sigma\lambda}\dot{q}_\lambda$

$\Rightarrow \underbrace{2\cdot\frac{1}{2}\sum_\sigma \sum_\lambda m_{\sigma\lambda}\dot{q}_\sigma \dot{q}_\lambda}_{2T} - T + V$

$= T + V = E = H$ $\square$

when $T$ is a quadratic form of the generalized coordinates $H$ is the total energy of the system

Let us make the earlier more transparent with an example:

2-variables

$$L = \underbrace{\frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2)}_{T} - V(x_1, x_2)$$, new coordinates $q_1$ & $q_2$ $\quad x_1 = x_1(q_1, q_2)$

$x_2 = x_2(q_1, q_2)$

then $\dot{x}_1 = \frac{\partial x_1}{\partial q_1}\dot{q}_1 + \frac{\partial x_1}{\partial q_2}\dot{q}_2$ & $\dot{x}_2 = \frac{\partial x_2}{\partial q_1}\dot{q}_1 + \frac{\partial x_2}{\partial q_2}\dot{q}_2$

so kinetic energy takes the form: $T = \frac{m}{2}(A\dot{q}_1^2 + B\dot{q}_1\dot{q}_2 + C\dot{q}_2^2)$

$$\Rightarrow H = \frac{\partial L}{\partial \dot{q}_1}\dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2}\dot{q}_2 - L = m(A\dot{q}_1 + \tfrac{B}{2}\dot{q}_2)\dot{q}_1 + m(\tfrac{B}{2}\dot{q}_1 + C\dot{q}_2)\dot{q}_2 - L$$

$$= 2T - T + V = T + V \quad \square$$

| H conserved / $H=E$ | YES | NO |
|---|---|---|
| YES | A | B |
| NO | C | D |

A) Example: $x = x(q)$, $\dot{x} = \frac{\partial x}{\partial q}\dot{q}$, $L = \frac{m}{2}\dot{x}^2 - V(x) \Rightarrow \frac{m}{2}\left(\frac{\partial x}{\partial q}\right)^2 \dot{q}^2 - V(q) = L$

$$p = \frac{\partial L}{\partial \dot{q}} = m\left(\frac{\partial x}{\partial q}\right)^2 \dot{q}$$

$$H = p\dot{q} - L = \frac{m}{2}\left(\frac{\partial x}{\partial q}\right)^2 \dot{q}^2 + V(q) = \frac{p^2}{2m} + V(x) = \text{energy}$$

B) $V = V(x,t)$ : $L = \frac{m}{2}\dot{x}^2 - V(x,t)$, $p = m\dot{x} \Rightarrow H = \frac{p^2}{2m} + V(x,t) = E$

but H is not conserved since there is an explicit t dependence.

C) Bead on a rotating rope: $x(t) = r(t)\cos\omega t$, $y(t) = r(t)\sin\omega t$
& potential $V(r)$

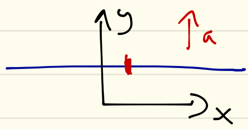
$$\dot{x} = \dot{r}\cos\omega t - r\omega\sin\omega t, \quad \dot{y} = \dot{r}\sin\omega t + r\omega\cos\omega t$$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(r) = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow H = p_r\dot{r} - L = \frac{p_r^2}{2m} - \frac{m}{2}r^2\omega^2 + V(r)$$

There is no explicit t dependence so H is conserved.
But it is not energy since there is a minus sign in $-\frac{m}{2}r^2\omega^2$

D) Accelerating rod with a bead:




$$\dot{y} = at, \quad y = at^2/2$$

$$L = \frac{m}{2}(\dot{x}^2 + (at)^2) - mg\left(\frac{at^2}{2}\right)$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow H = p\dot{x} - L = \frac{p^2}{m} - \frac{p^2}{2m} - \frac{m}{2}a^2t^2 + mg\frac{at^2}{2}$$

H not conserved since explicit time dependence
Also not E since $-\frac{m}{2}a^2t^2$ has a minus sign.

## Hamilton's equations:

For $N$ degrees of freedom we have $H = \left(\sum_{\sigma=1}^{N} p_\sigma \dot{q}_\sigma(q,p)\right) - L(q, \dot{q}(q,p))$, $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$

where we used shorthand $(q,p) = (q_1 \dots q_N, p_1 \dots p_N)$

For clarity assume just 1D problem. ($N>1$ goes similarly)

Let us see what $\frac{\partial H(q,p)}{\partial p}$ is

note the $q,p$ dependence possibility

$$\frac{\partial H}{\partial p} = \underbrace{\frac{\partial\, p\dot{q}(q,p)}{\partial p}}_{(*)} - \underbrace{\frac{\partial L(q, \dot{q}(q,p))}{\partial p}}_{(**)}$$

$(*) \Rightarrow \dot{q} + p\frac{\partial \dot{q}}{\partial p}$

$(**) \Rightarrow \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}(q,p)}{\partial p}$ where $\frac{\partial L}{\partial \dot{q}} = p$

$$\Rightarrow \boxed{\frac{\partial H}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - p\frac{\partial \dot{q}}{\partial p} = \dot{q}}$$

what about $\frac{\partial H}{\partial q} = \underbrace{\frac{\partial\, p\dot{q}(q,p)}{\partial q}}_{p\frac{\partial \dot{q}}{\partial q}} - \underbrace{\frac{\partial L(q, \dot{q}(q,p))}{\partial q}}_{\frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} = \frac{\partial L}{\partial q} + p\frac{\partial \dot{q}}{\partial q}}$

Lagrange equation: $\dot{p} = \frac{\partial L}{\partial q}$

$$\Rightarrow \boxed{\frac{\partial H}{\partial q} = -\dot{p}}$$

we have a pair of 1st order differential equations

$$\dot{q} = \frac{\partial H}{\partial p} \quad \& \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Result generalizes to many variables:

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad \& \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Example: harmonic oscillator $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \& \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

Hamilton's equations via modified Hamilton's principle: (assume 1D for clarity)

$$L = p\dot{q} - H$$

$$\Rightarrow S = \int_{t_1}^{t_2} L\,dt = \int (p\dot{q} - H)\,dt \qquad \left(\text{integrand} = \left[p(t)\frac{dq(t)}{dt} - H(q(t), p(t))\right] dt\right)$$

Hamilton's principle: action at extremum with respect to variations in both $q$ & $p$.

$p(t) \to p(t) + \delta p(t)$ & $q(t) \to q(t) + \delta q(t)$, $\delta p(t_1) = \delta p(t_2) = 0$ & $\delta q(t_1) = \delta q(t_2) = 0$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \Big(p \underbrace{\delta\dot{q}}_{\frac{d}{dt}\delta q} + \dot{q}\,\delta p - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p\Big)\,dt = 0$$

$$= \underbrace{\Big/_{t_1}^{t_2} p\,\delta q}_{0} + \int_{t_1}^{t_2} \left[\left(-\dot{p} - \frac{\partial H}{\partial q}\right)\delta q + \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p\right] dt$$

$\delta p$ & $\delta q$ variations are independent $\Rightarrow$ $\dot{q} = \frac{\partial H}{\partial p}$ & $\dot{p} = -\frac{\partial H}{\partial q}$

... generalize to $H = \sum_\sigma p_\sigma \dot{q}_\sigma - L(q_1 \dots q_n, \dot{q}_1(q,p) \dots \dot{q}_n(q,p))$

by independent variations $\delta q_\sigma$ & $\delta p_\sigma$.

More carefully

$$S+\delta S = \int \Big[ \underbrace{(p+\delta p)(\dot q+\delta \dot q)}_{(*)} - \underbrace{H(q+\delta q, p+\delta p)}_{(**)} \Big] dt$$

$(*) \to$ 1st order in $\delta$: $\delta p \dot q + p \frac{d}{dt}\delta q$

$(**) \Rightarrow$ 1st order in variations using Taylor expansion

$$\frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p$$

$\Rightarrow$ Then follow the steps on previous page.

The Book derived also like this:

$$H = \sum_\sigma p_\sigma \dot q_\sigma - L \Rightarrow dH = \sum_\sigma \dot q_\sigma dp_\sigma + p_\sigma d\dot q_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot q_\sigma} d\dot q_\sigma - \frac{\partial L}{\partial t} dt$$

Cancel

$$\Rightarrow \sum_\sigma \left( \dot q_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma \right) - \frac{\partial L}{\partial t} dt = dH$$

on the other hand: $dH = \sum_\sigma \frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma + \frac{\partial H}{\partial t} dt$

$\Rightarrow \dot q_\sigma = \frac{\partial H}{\partial p_\sigma}$ & $-\frac{\partial L}{\partial q_\sigma} = \frac{\partial H}{\partial q_\sigma}$, but Lagrange equation implies $\frac{\partial L}{\partial q_\sigma} = \dot p_\sigma$

$\Rightarrow$ Hamilton's equations

$$\dot q_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\dot p_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

## Example: 2D + central forces

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \quad \begin{cases} x = r\cos\theta, & \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ y = r\sin\theta, & \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$$

$$\Rightarrow T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Leftrightarrow \dot{r} = \frac{p_r}{m} \quad \& \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Leftrightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = p_r\dot{r} + p_\theta \cdot \dot{\theta} - T + V = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

$$\begin{cases} \dot{r} = \frac{\partial H}{\partial p_r} = p_r/m \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = p_\theta/mr^2 \\ \dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{\partial V}{\partial r} - \frac{p_\theta^2}{m}\frac{1}{r^3} \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \end{cases}$$

$$\dot{\theta} = L/mr^2$$

$$\dot{p}_r = -\frac{\partial V}{\partial r} - \frac{L^2}{mr^3}$$

Last one: $mr^2\dot{\theta} = L = p_\theta$

## Symmetries for time translation?

Let us say we shift time coordinate $t' = t + dt \Rightarrow t = t' - dt$

$$L(q(t), \dot{q}(t), t) \to L(q(t+dt), \dot{q}(t+dt), t+dt)$$

$$q(t+dt) \approx q(t) + dt\,\dot{q}(t), \quad \dot{q}(t+dt) = \dot{q}(t) + dt\,\ddot{q}$$

Lagrange eq. for the 1st term

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) + \frac{\partial L}{\partial t},$$ if $L$ has no explicit time dependence $\partial L/\partial t = 0$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right] = 0 \qquad H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = \text{conserved} = \text{Hamiltonian}$$

generalize to many generalized coordinates $\Rightarrow H = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L$

$$L = \frac{m}{2}x^2 - V(x) \Rightarrow H = \frac{\partial L}{\partial \dot{x}}\dot{x} - L = \frac{m}{2}\dot{x}\cdot\dot{x} - \frac{m}{2}\dot{x}^2 + V(x) = \frac{m}{2}\dot{x}^2 + V(x) = \text{energy}$$

# Example on Symmetries:

Lets inspect symmetries of the Lagrangian

$$L = \frac{m}{2}\left(4\dot{x}^2 - 2\dot{x}\dot{y} + 2\dot{y}^2\right) + C(2x-y)$$

Now $\begin{cases} x \to x+\varepsilon \\ y \to y+2\varepsilon \end{cases}$ $\Rightarrow$ Potential term $C(2x-y)$ is clearly invariant (actually to all orders of $\varepsilon$)

Also $\dot{x} \to \dot{x} + \underbrace{\frac{d}{dt}\varepsilon}_{0} = \dot{x}$ $\Rightarrow$ Also kinetic part invariant.

$\dot{y} \to \dot{y}$

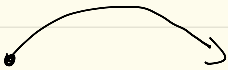
Symmetry transformation had $K_x = 1$ & $K_y = 2$

Conserved momentum

$$P(x,y,\dot{x},\dot{y}) = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = 1\cdot(4m\dot{x} - m\dot{y}) + 2(2m\dot{y} - m\dot{x})$$

$$= 2m\dot{x} + 3m\dot{y}$$

Another one: $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$



This is invariant under $x \to x+\varepsilon$ ($x$ = cyclic) and $y \to y+\varepsilon$

These have conserved momenta $P_x = 1\cdot\frac{m}{2}\cdot 2\dot{x} = m\dot{x}$ & $P_y = m\dot{y}$.

For 1st symmetry $(K_x, K_y, K_z) = (1,0,0)$ & for the 2nd

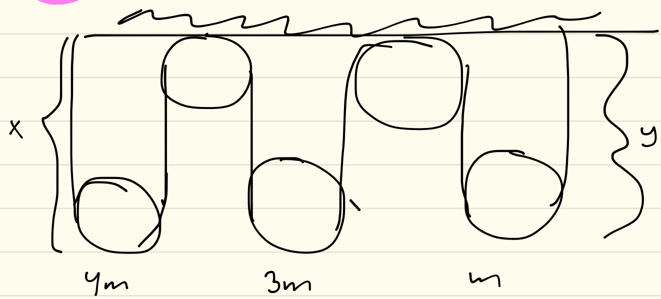
$(K_x, K_y, K_z) = (0,1,0)$

We could also look at the combination

$$\begin{cases} x \to x+\varepsilon \\ y \to y+\varepsilon \\ z \to z \end{cases} \Rightarrow (K_x, K_y, K_z) = (1,1,0) \Rightarrow P_{xy} = m\dot{x} + m\dot{y} = \text{conserved}.$$

Note: when $K_\alpha$ is a function of generalized coordinates kinetic term might not transform this easily. Here $K_\alpha$ was just constant.

Example: Atwood machine (more complicated)




Middle mass $-x-y$ to keep rope length the same

$$L = \frac{4m}{2}\dot{x}^2 + \frac{3m}{2}(-\dot{x}-\dot{y})^2 + \frac{m}{2}\dot{y}^2 + 4mgx + 3mg(-x-y) + mgy$$

$$= \frac{7}{2}m\dot{x}^2 + 3m\dot{x}\dot{y} + 2m\dot{y}^2 + mg(x-2y)$$

Transformation: $x \to x + 2\varepsilon$, $y \to y + \varepsilon$ leaves $L$ invariant $K_x = 2$, $K_y = 1$

$\Rightarrow$ conserved $P = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = m(17\dot{x} + 10\dot{y})$ $\square$